１次元間欠写像における長時間平均のふるまい

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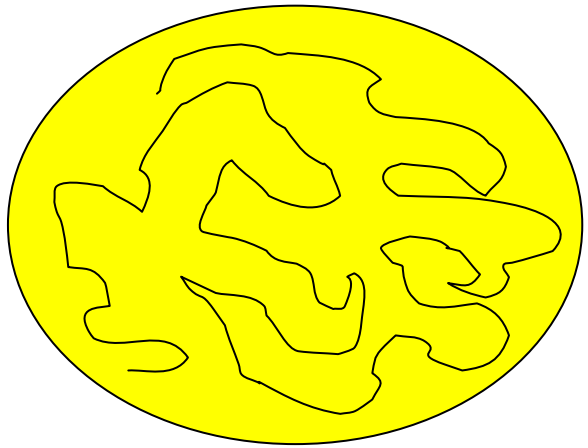
What is ergodicity of the non-equilibrium state?

1. Introduction (Ergodic problems and Intermittent phenomena)

2. Universal Distributions
   (Mittag Leffler, Generalized Arcsine, Stable)

3. Concluding Remarks



第3回　九州大学　産業技術数理研究センターワークショップ（兼　第3回連成シニュレーションフォーラム）　「自然現象における階層構造と数理的アプローチ」　2008年3月6日

Ergodic Theory

Birkhoff's ergodic theorem. (1931) Let T be measure preserving transformation of the probability space $(X, \mathcal{B}, m)$, then

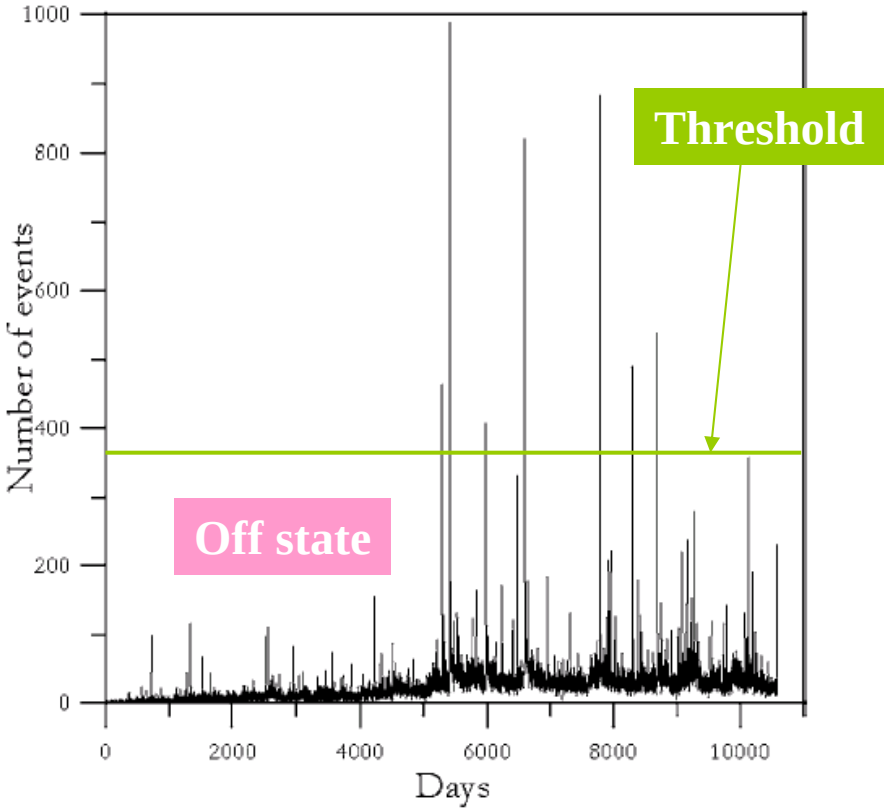
$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to \int_X f\,dm \quad \text{a.e. } \forall f \in L^1(m).$$

Question

$$\boxed{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to ? \quad \text{a.e. } \forall f \notin L^1(m).}$$

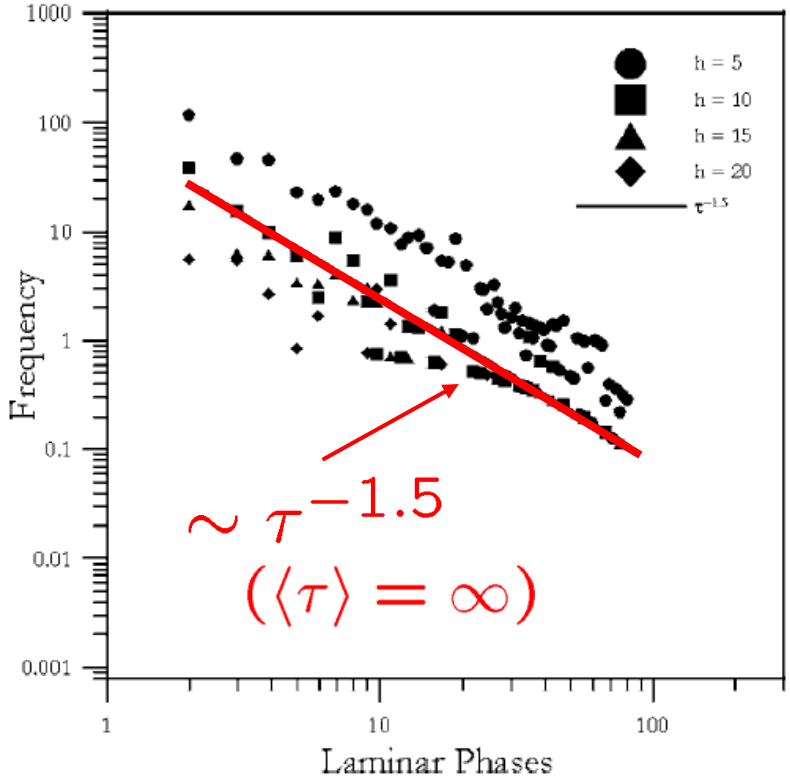
Intermittent Phenomena

M. Bottiglieri and C. Godano, **On-off intermittency in earthquake occurrence**, *Phys.Rev*. E **75**, 026101 (2007).



$\Delta t = 1$ (day)

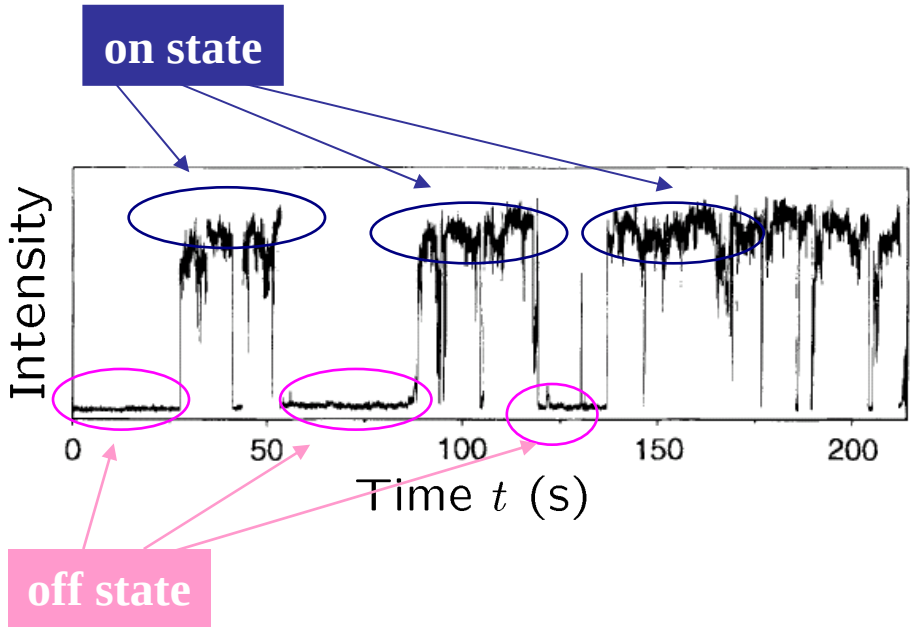
(the Southern California catalog, 1973-2003)

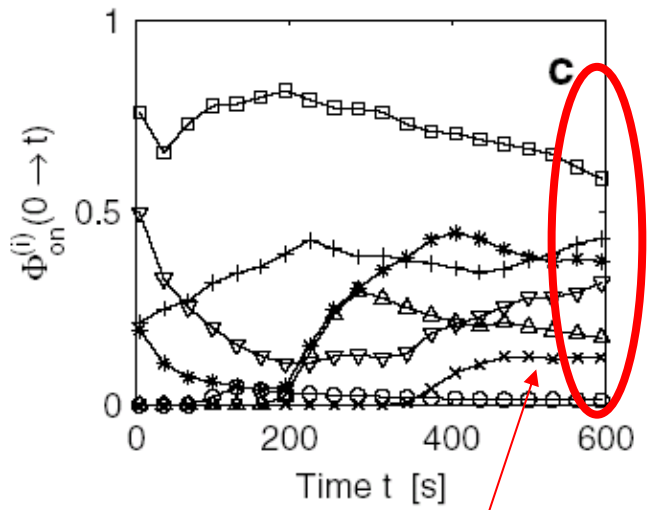


$\Delta t = 600$ (min)

Nonergodicity

X.Brokmann et al., **Statistical aging and nonergodicity in the fluorescence of single nanocrystals**, *Phys. Rev. Lett.* **90**, 120601 (2003).





Characteristics

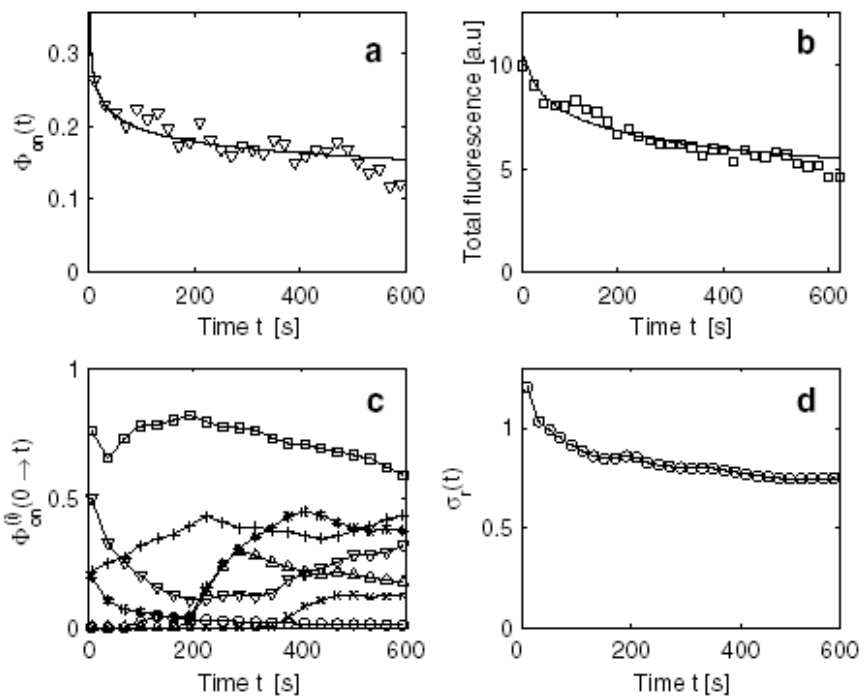
Power law, 1/f spectrum,

Non-stationarity, Non-ergodicity

The ratio of the on state time $\Phi_{on}(t)$ does not converge.

Nonergodicity and Non-stationarity

Nonstationary and non-ergodic behavior



the fraction of QDs in the on state

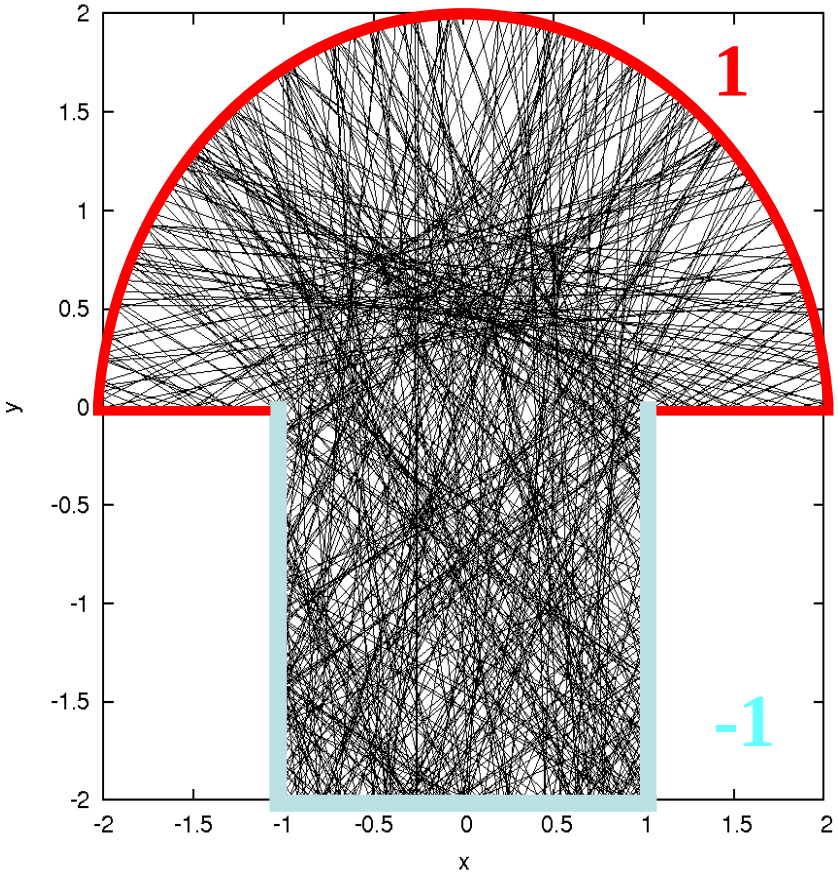
$$\Phi_{\text{on}}(t) = \frac{1}{215}\sum_{k=1}^{215} 1_{[I_0,\infty)}(I_k(t)) \sim t^{-\beta_1}$$

the fraction of time in the on state (for the 7th QD)

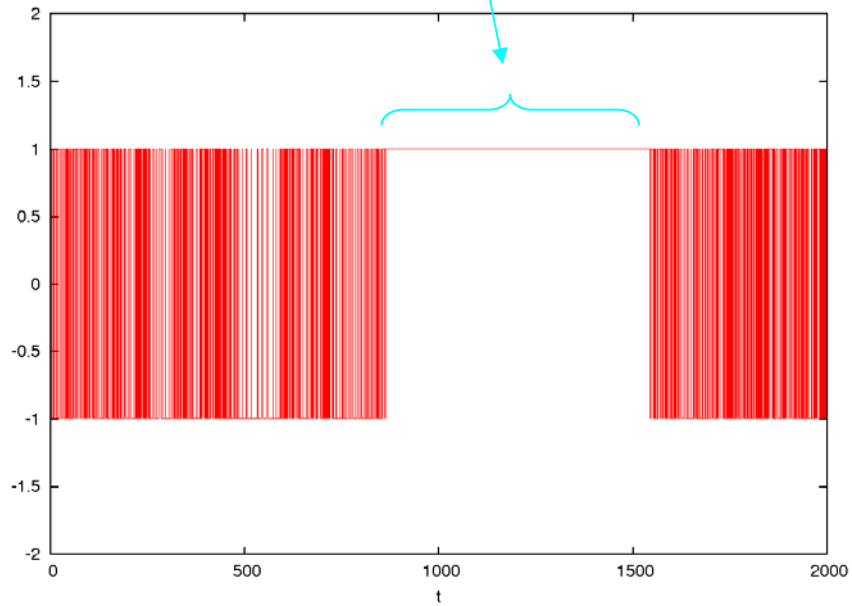
$$\Phi_{\text{on}}^{(i)}(t) = \frac{T_{\text{on}}^{(i)}(t)}{t}$$

Time average does not converge to constant value.

Mushroom Billiard

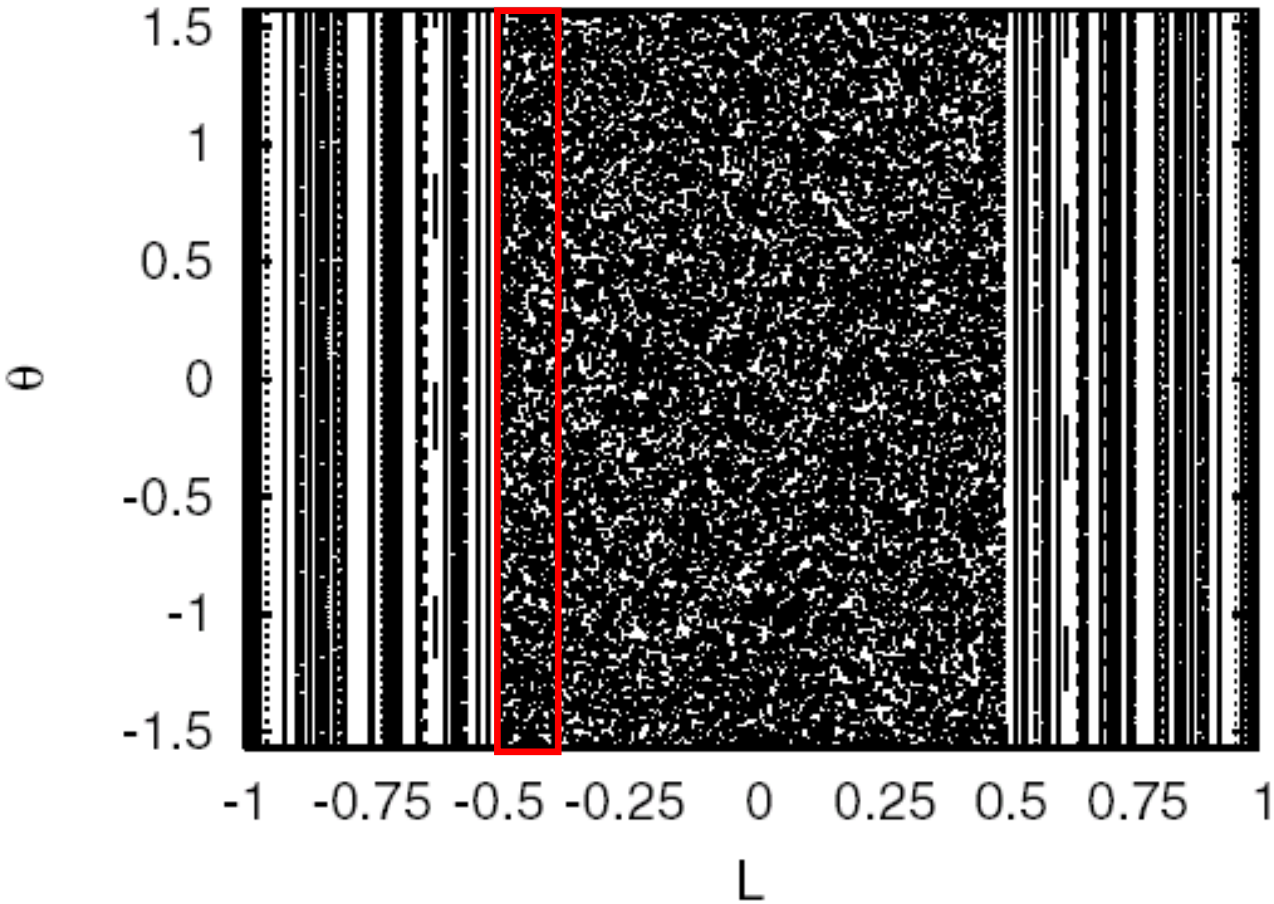


$$P(n) \propto n^{-3} \ (\langle n \rangle < \infty)$$



T. Miyaguchi, **Escape time statistics for mushroom billiard**, *Phy. Rev.* **E 75** 066215 (2007). (Aizawa lab. Waseda univ. Satoru Tsugawa)

Remark



Survival probability $\Rightarrow P(n) \propto n^{-2}$ $(\langle n \rangle = \infty)$

Purpose

Analyzing the distribution of the time average of some observation functions in infinite measure systems which are related to intermittent phenomena.

To make clear the non-stationary phenomena and

the foundation of the ergodicity in the non-equilibrium state

Conditions

Let c denote the inner endpoint of the monotonicity intervals, and let $I_0 = (0, c), I_1 = (c, 1)$. Then T is assumed to satisfy the following conditions:

(i) $T_{|I_j}$ has a C^1-extension $\overline{T}_{|I_j}$ to $\overline{I}_j$, $(\overline{T}_{|I_j})' > 0$, and $\overline{T(I_j)} = [0, 1]$ $(j = 1, 2)$.

(ii) T is convex in a neighbourhood of 0.

(iii) T admits an invariant measure μ equivalent ot λ, such that the density $dmu/d\lambda$ has a version of the form

$$h(x) = \tilde{h}(x)/x^p, \quad x \in (0, 1], \text{ where } p \geq 1,$$

and $\tilde{h}$ is positive, continuous and of bounded variation on $[0, 1]$.

(iv) The function $\psi = \dfrac{h \circ T \cdot T'}{h}$ is increasing on I_0.

Infinite measure systems

♠ *Lasota-Yorke map* T is defined on $[0,1]$ as

$$Tx = \begin{cases} \frac{x}{x-1} & x \in [0, 1/2) \\ 2x - 1 & x \in [1/2, 1], \end{cases}$$

which has invariant density $\rho(x) = 1/x$, and $\boxed{Tx - x \sim x^2 \ (x \to 0)}$.

♠ Boole transformation $T : \mathbb{R} \to \mathbb{R}$ is defined as

$$Tx = x - 1/x$$

which has uniform invariant density.

Invariant density can not be normalized.

♠ *Log-Weibull map*

$$Tx = \begin{cases} x + x^2 e^{-1/x} & x \in [0, a] \\ (x-a)/(1-a) & x \in (a, 1]. \end{cases}$$

The invariant density has an **essential singularity** at 0.

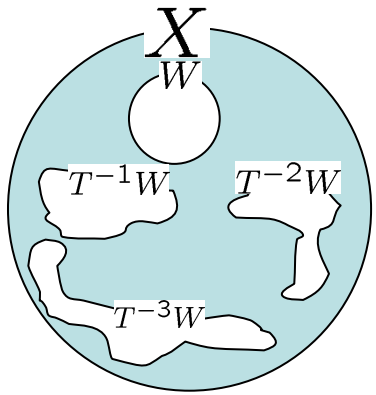
$$\boxed{\rho(x) = g(x) e^{1/x}/x, \quad g \text{ continuous and positive on } [0,1].}$$

The residence time distribution obeys the Log-Weibull distribution.

Conservative and Dissipative

Def. **Wandering set**

A set $W \subset X$ is called a *wandering set* if the sets $\{T^{-n}W\}_{n=0}^{\infty}$ are disjoint.



$\mathcal{D}(T)$ (dissipative part): a coutable union of wandering sets
$\mathcal{C}(T)$ (conservative part): $\mathcal{C}(T) = X \setminus \mathcal{D}(T)$

Def. **Conservative and Dissipative**

The transformation T is called *conservative* if $\mathcal{C}(T) = X \mod \mu$,
and *dissipative* if $\mathcal{D}(T) = X \mod \mu$.

Remark

$\mu(X) < \infty \Rightarrow$ consevative

DKA Limit Theorem

Darling – Kac – Aaronson Limit Theorem (1981)

If T is a conservative, ergodic measure preserving transformation of $(X, \mathcal{B}, m)$, then $\exists\ a_n$ s.t. $\forall f \in L^1_+(m)$

$$\frac{1}{a_n}\sum_{k=0}^{n-1} f(T^k \cdot) \to M_\alpha \quad \text{as } n \to \infty.$$

Random variable (Mittag-Leffler distribution)

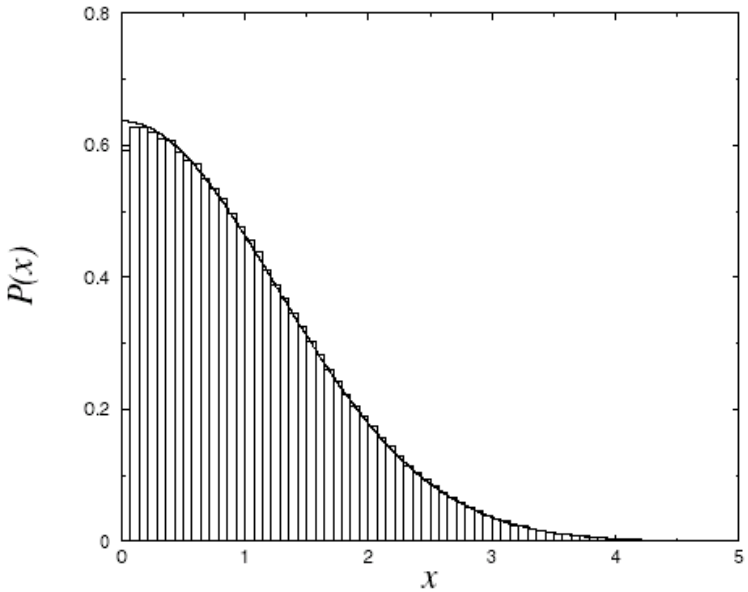
$$f > 0 \text{ and } \int_X |f| dm < \infty$$

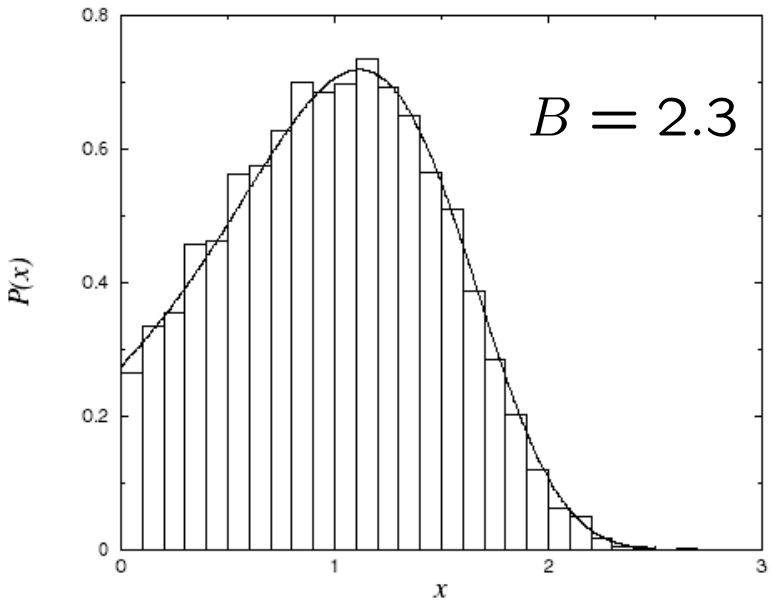
$$E(e^{-zM_\alpha}) = \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha)^k (-z)^k}{\Gamma(1+k\alpha)}$$

Lyapunov exponent

$$f(x) = \log |(Tx)'|$$

Mittag-Leffler Distribution





Lyapunov exponent for Boole transformation

$$Tx = x - \frac{1}{x}$$

$$\left(\text{return sequence} : a_n = \frac{n^{1/2}}{\sqrt{2\pi}}\right)$$

$$\boxed{f(x) = \log|(Tx)'|}$$

Lyapunov exponent for MB map

$$Tx = \begin{cases} x + 2^{B-1}x^B & x \in [0, 1/2] \\ x - 2^{B-1}(1-x)^B & x \in (1/2, 1] \end{cases}$$

$$\left(\text{return sequence} : a_n \propto n^{\beta-1}\right)$$

Skew Modified Bernoulli Map

The skew modified Bernoulli map is closely related to the intermittent phenomena.
(Rayleigh-Benard convection, Lorentz model)

Skew Modified Bernoulli map

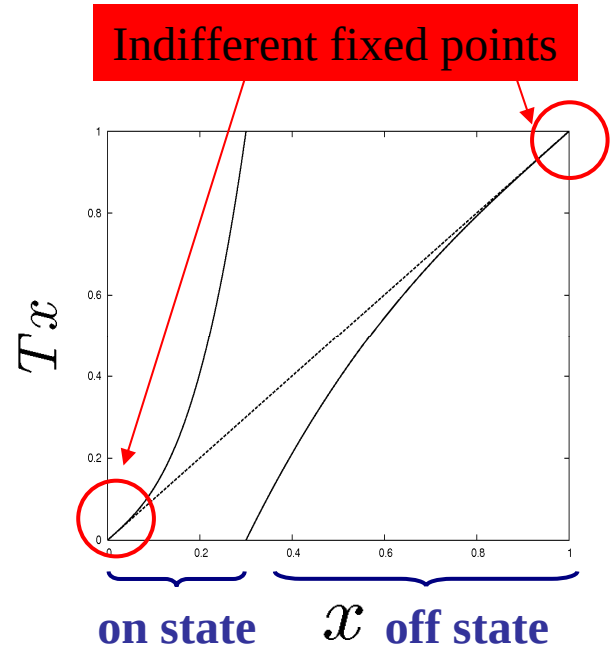
$$Tx = \begin{cases} x + (1-c)\left(\frac{x}{c}\right)^B & x \in [0, c] \\ x - c\left(\frac{1-x}{1-c}\right)^B & x \in (c, 1] \end{cases}$$

Invariant measure (Infinite measure)

$$\rho(x) \propto x^{1-B} + (1-x)^{1-B}$$

$B < 2$: finite measure

$B \geq 2$: infinite measure



MB map (B=3.0,c=0.3)

Lamperti-Thaler Generalized Arcsine Law

Let T be the skew modified Bernoulli map, then

$$\lim_{n\to\infty} \Pr\left(\frac{1}{n}\sum_{k=0}^{n-1} 1_{[0,c]} \circ T^k \le x\right) = G_{\alpha_1,\alpha_2}(x),$$

where $\alpha_1 = \beta - 1$,

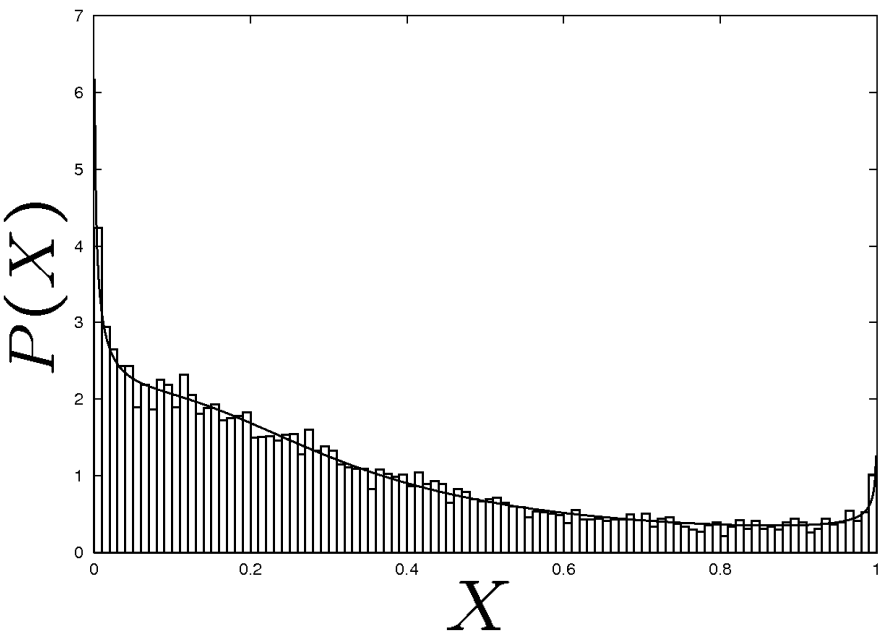
$$\alpha_2 = \frac{1+(B-1)c}{1+(B-1)(1-c)}\left(\frac{1-c}{c}\right)^{\frac{2}{B-1}}$$

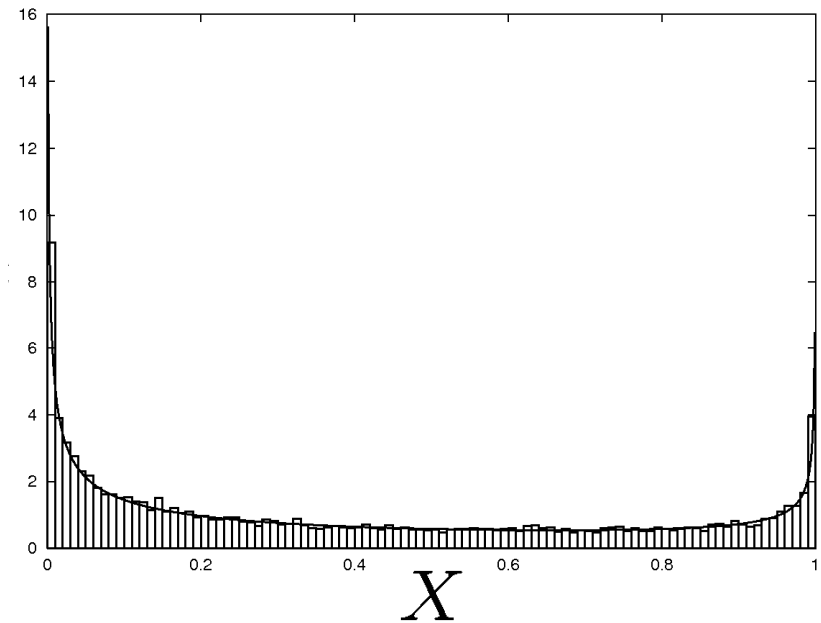
and the p.d.f. $G'_{\alpha_1,\alpha_2}(x)$ *is given by*

$$G'_{\alpha_1,\alpha_2}(x) = \frac{\alpha_2 \sin \pi\alpha_1}{\pi} \frac{x^{\alpha_1-1}(1-x)^{\alpha_1-1}}{\alpha_2^2 x^{2\alpha_1} + 2\alpha_2 x^{\alpha_1}(1-x)^{\alpha_1}\cos\pi\alpha_1 + (1-x)^{2\alpha_1}}$$

M. Thaler, **A limit theorem for sojourns near indifferent fixed points of one-dimensional maps**, *Ergod. Th. & Dynam. Sys.* 22 1289-1312 (2002).

Generalized Arcsine Law



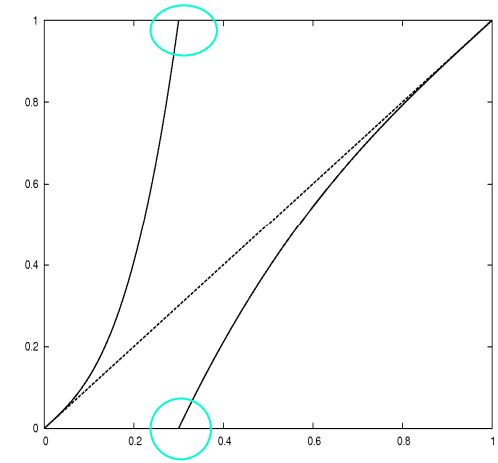


Probability density function of $X = \frac{1}{n}\sum_{k=0}^{n-1} 1_{[0,c]} \circ T^k$

Remark on the invariant density and mean

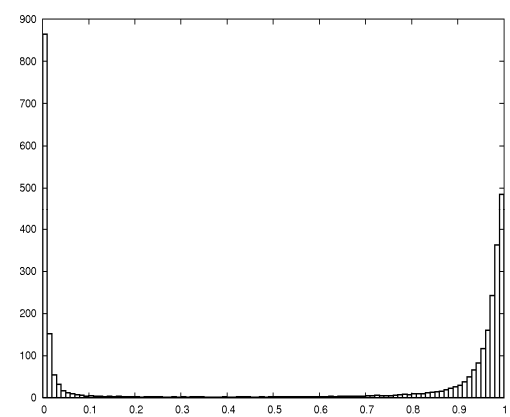
$$P_+(n_+) \propto \left(\frac{n_+}{c_+}\right)^{-\frac{B}{B-1}}$$ **(On state)**

$$P_-(n_-) \propto \left(\frac{n_-}{c_-}\right)^{-\frac{B}{B-1}}$$ **(Off state)**



$$\frac{c_+}{c_-} \propto \frac{\langle n_+ \rangle}{\langle n_- \rangle} = q(c) = \frac{1-\alpha_2}{\alpha_2}$$

$q(c) \neq 1$ $(c \neq 1/2)$ and $q'(c) > 0$



The invariant density is not symmetric.

Universal Distributions

Time Average of the observation function

$$\frac{1}{a_n}\sum_{k=0}^{n-1} f\circ T^k = \frac{f + f\circ T + \cdots + f\circ T^{n-1}}{a_n} \to \mathbf{X} \text{ as } n \to \infty$$

where $a_n = n^\gamma$.

Random variables

Distributional Limit Theorems for the Time Average

$f(x)$	invariant measure	γ	distribution of $\mathbf{X}$
L^1_m	**finite**	1	*delta* (Birkhoff, 1931)
L^1_m	**infinite**	$\frac{1}{B-1}$	*Mittag-Leffler* (Aaronson, 1981)
non-L^1_m	**finite**	$\frac{\alpha}{2-B}$	*Stable* (**Akimoto [1]**)
$L^1_{\mathrm{loc}}(m,(0,1))$	**infinite**	1	*Generalized Arcsine* (**Akimoto [1]**)
non-L^1_m	**infinite**	$\frac{\alpha}{B-1}+1$	*Stable* (**Akimoto [1]**)

[1] T. Akimoto, Generalized Arcsine Law and Stable Law in an Infinite Measure System, arXiv:0801.1382v.

$L^1_{\text{loc},m}(\Omega)$ with finite mean

Definition

Locally integrable

Finite mean

A function $f(x)$ is said to be *locally integrable with finite mean with respect to the measure* m on Ω provided that $f \in L^1_m(K)$ $\forall$ compact set $K \subset \Omega$: $\boxed{\int_K |f(x)|dm < \infty}$; and $\boxed{\lim_{\epsilon \to 0} \frac{\int_\epsilon^{1-\epsilon} f dm}{\int_\epsilon^{1-\epsilon} dm} < \infty}$.

In this case we call f *the* $L^1_{\text{loc},m}(\Omega)$ *function with finite mean*.

Examples in the MB map

♠ $f(x) = \begin{cases} 1 & x \in [0, 1/2) \\ -1 & x \in [1/2, 1) \end{cases}$

♠ $f(x) = x$

are the $L^1_{\text{loc},m}(0,1)$ function with finite mean. (non-L^1_μ functions).

Generalized Arcsine Law

Let T be the skew MB map and f be the $L^1_{\text{loc},m}(0,1)$ function with finite mean and $f(0) = a, f(1) = b$. Further, there exists δ such that $0 < \delta < 1$ and $f(x)$ is continuous in $[0,\delta] \cup [1-\delta, 1]$. Then the time average converges in distribution to $Y_{\alpha_1,\alpha_2,a,b}$:

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k \cdot) \to Y_{\alpha_1,\alpha_2,a,b} \text{ as } n \to \infty$$

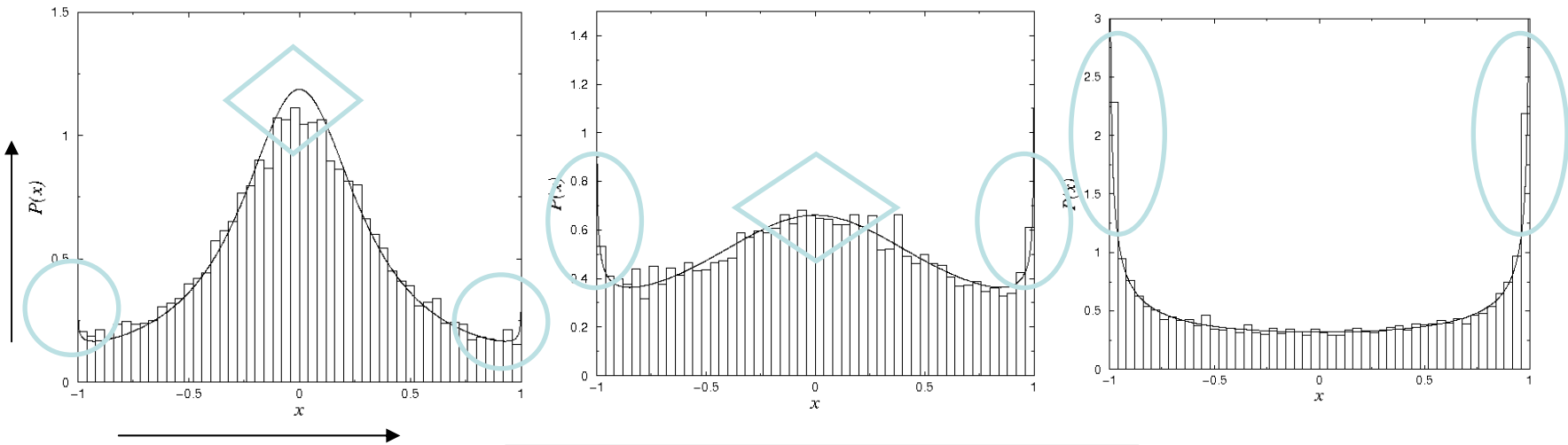
where $\gamma = 1/(B-1)$ and the p.d.f. of $Y_{\alpha_1,\alpha_2,a,b}$ for $a > b$ is given by

Generalized Arcsine distribution

Numerical Simulations $(c = 1/2)$

$B = 2.2$ $(\alpha_1 \cong 0.83)$ $B = 2.4$ $(\alpha_1 \cong 0.71)$ $B = 3.0$ $(\alpha_1 = 0.5)$

p.d.f. of the time average



Time average

Distributions of the time average

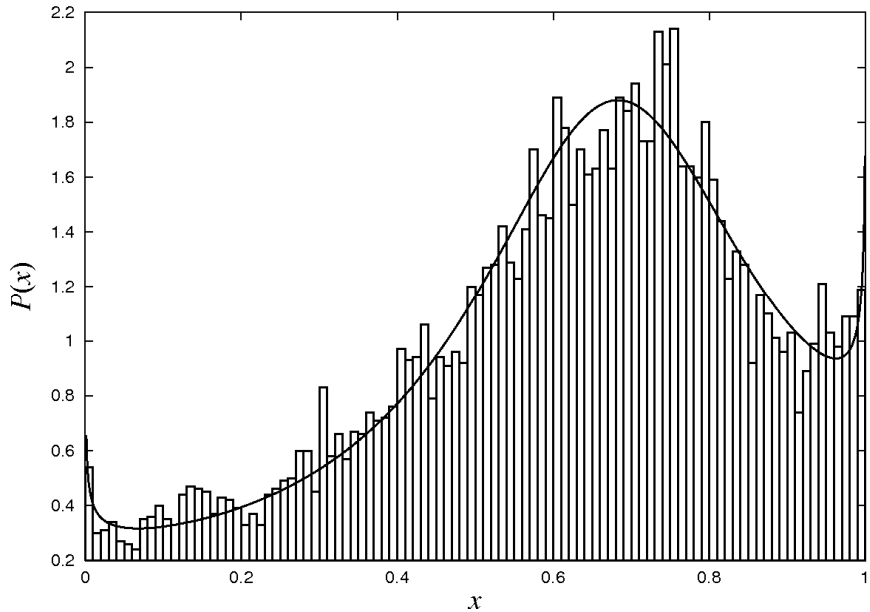
Observation function

$$f(x) = \begin{cases} 1 & (x \leq \frac{1}{2}) \\ -1 & (x > \frac{1}{2}) \end{cases}$$

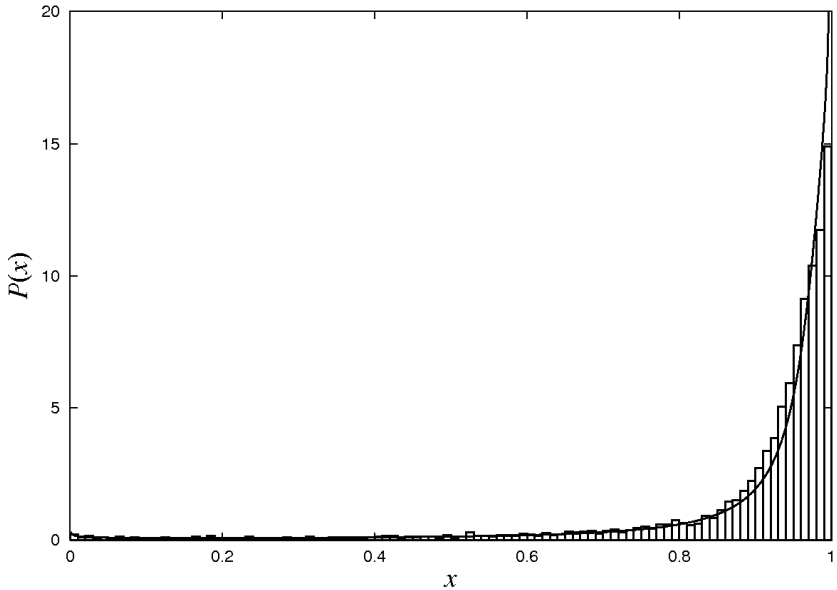
As the value of B becomes large, the middle peak becomes low and edge peaks become high.

Numerical Simulations

$B = 2.3$ and $c = 0.4$



$B = 2.5$ and $c = 0.1$



Probability density function of $x = \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$

$(f(y) = y^2)$

Application to Correlation Function

For all $n \geq 0$

$$C(n) = \frac{1}{N}\sum_{k=0}^{N-1} f(x_k)g(x_{k+n}) \to Y_{\alpha_1,\alpha_2,a,b} \quad N \to \infty$$

where $f \cdot g \circ T^n \in L^1_{\mathrm{loc},m}(0,1)$ with finite mean and $f(0)g(0) = a, f(1)g(1) = b$.

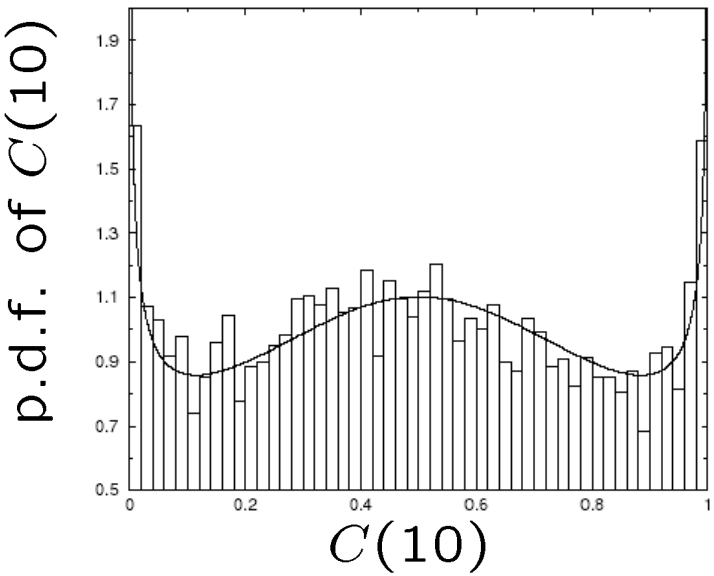
Correlation function is intrinsically random (Generalized Arcsine distribution) and never decays.

Remark. The convergence becomes slow as n becomes large.

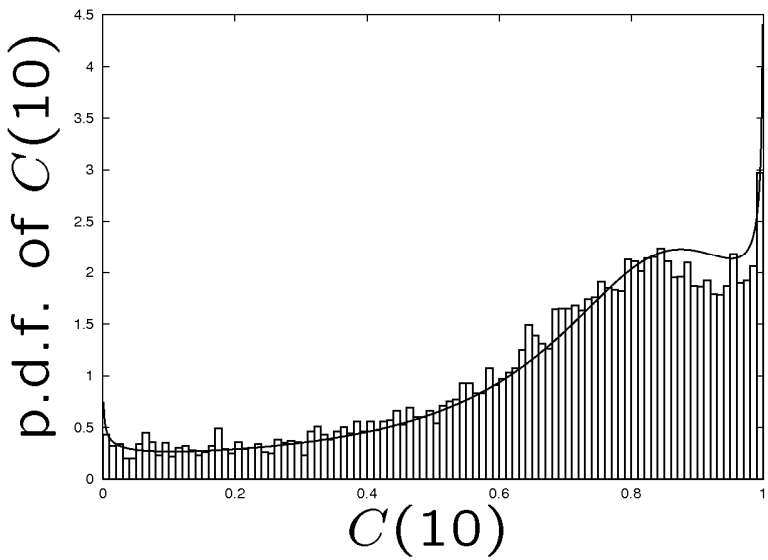
$$\left(C(n) = \frac{1}{N}\sum_{k=0}^{N-1} x_k x_{k+n} \right)$$

Correlation Function

$B = 2.5$ and $C = 0.5$



$B = 2.4$ and $C = 0.4$



Probability density function of $C(10) = \frac{1}{n}\sum_{k=0}^{n-1} x_k x_{k+10}$

Remark on Wiener-Khintchine Theorem

Wiener-Khintchine Theorem

$$S(k,N) \cong \sum_{n=1}^{N} C(n)\cos(2\pi kn/N)$$

where $C(n)$ is the correlation function. In the case of the power law decay, namely, $C(n) \sim n^{-\alpha}$, there is a posibility that $S(k,N)$ diverges. Actually, $S(k,N)$ diverges when the exponent α is smaller than 1.

$$\begin{aligned} S(k,N) &\cong \sum_{n=1}^{N} n^{-\alpha}\cos(2\pi kn/N) = N^{1-\alpha}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{n}{N}\right)^{-\alpha}\cos\left(2\pi k\frac{n}{N}\right) \\ &\sim c(k)N^{1-\alpha} \cong k^{\alpha-1}N^{1-\alpha} \quad \text{as } N \to \infty \end{aligned}$$

where $c(k) = \int_0^1 x^{-\alpha}\cos(2\pi kx)dx = k^{\alpha-1}\frac{1}{(2\pi)^{-\alpha}}\int_0^1 x^{-\alpha}\cos x dx < \infty$.

Power Spectrum

$$S_k = |\hat{x}_k|^2 = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{l=1}^{N} x_j x_l \cos\left(\frac{2\pi k}{N}(j-l)\right)$$

$C(n)$:random $\Rightarrow$ S_k : random

Question

1. The distribution of S_k
2. Scaling of S_k ($S_k \sim N^{-\nu}$)

Distribution of S_0

$$S_0 = \frac{1}{N^2}\sum_{j=1}^{N}\sum_{l=1}^{N} x_j x_l = \underline{\left(\frac{1}{N}\sum_{j=1}^{N} x_j\right)}\ \underline{\left(\frac{1}{N}\sum_{l=1}^{N} x_l\right)}$$

$$\rightarrow (Y_{\alpha_1,\alpha_2,0,1})^2$$

Generalized arcsine distribution

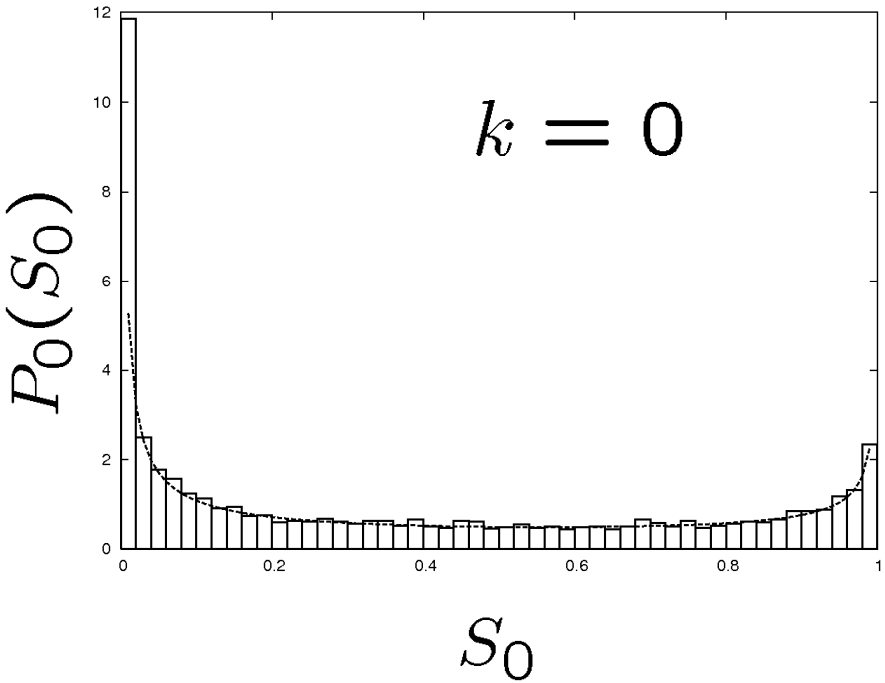
The probability density function of $x = S_0$ is given by

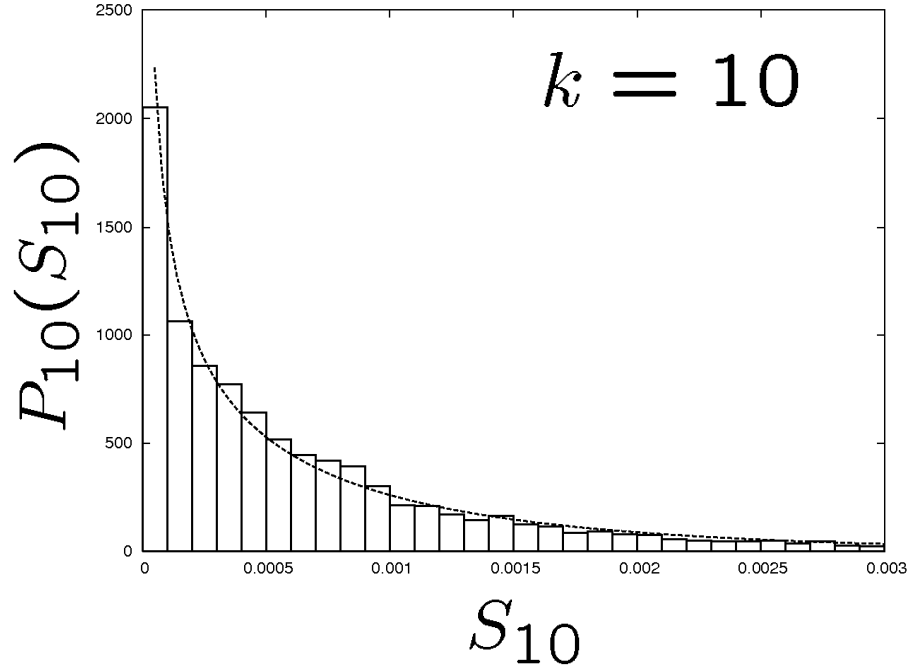
$$P_0(x) = \frac{1}{2\sqrt{x}} G'_{\alpha_1,\alpha_2}(\sqrt{x})$$

$$\left[\Pr(S_0 < x) = \Pr(Y_{\alpha_1,\alpha_2,0,1} < \sqrt{x})\right]$$

Numerical Simulations

$B = 3.0$ and $c = 0.5$





$$P_0(x) = \frac{1}{2\pi\sqrt{x}\sqrt{\sqrt{x}(1-\sqrt{x})}}$$

$$P_{10}(x) = Ax^{-0.48}e^{-750x}$$

(Gamma distribution)

$L^1_{\mathrm{loc},m}(\Omega)$ with infinite mean

Definition

Locally integrable

Infinite mean

A function $f(x)$ is said to be *locally integrable with finite mean with respect to the measure* μ on Ω provided that $f \in L^1_\mu(K)$ $\forall$ compact set $K \subset \Omega$: $\boxed{\int_K |f(x)|dm < \infty}$; and $\boxed{\lim_{\epsilon\to 0} \frac{\int_\epsilon^{1-\epsilon} f dm}{\int_\epsilon^{1-\epsilon} dm} = \infty}$.

In this case we call f *the* $L^1_{\mathrm{loc},m}(\Omega)$ *function with infinite mean*.

$$\frac{1}{a_n}\sum_{k=0}^{n-1} f\circ T^k \to ? \quad \text{a.e. } \forall f \in L^1_{loc,m}(0,1) \text{ with infinite mean.}$$

Stable Distributions

Theorem 1. For fixed $0 < \alpha < 1$ the function $\gamma_\alpha(\lambda) = e^{-\lambda^\alpha}$ is the Laplace transform of a distribution G_α with the following properties:

G_α is stable; more precisely, if $\mathbf{X}_1, \cdots, \mathbf{X}_n$ are independent variables with the distribution G_α, then $(\mathbf{X}_1 + \cdots + \mathbf{X}_n)/n^{1/\alpha}$ has again the distribution G_α.

$$x^\alpha[1 - G_\alpha(x)] \to \frac{1}{\Gamma(1-\alpha)} \quad x \to \infty,$$

$$e^{x^{-\alpha}} G_\alpha(x) \to 0 \quad x \to 0.$$

<u>Power law phenomena</u>

Earthquake, fluorescence intermittency of nanocrystals, motion of bacteria, chaotic dynamics, finance

Theorem and Conjecture

Finite measure case $(B < 2)$

$$\frac{2}{\Gamma(1-\alpha)n^{1/\alpha}} \sum_{k=0}^{n-1} f \circ T^k \to G_\alpha \quad n \to \infty,$$

where $f(x) = x^{-\beta}$ $(\beta \geq 2 - B)$ and $\alpha = \beta/(2-B)$.

Infinite measure case $(B \geq 2)$

$$\frac{1}{b_n} \sum_{k=0}^{n-1} f \circ T^k \to G_{1/\gamma} \quad n \to \infty,$$

where $b_n \propto n^\gamma$ and $\gamma = \frac{\beta}{B-1} + 1$.

$x^\beta f(x) = O(1), \quad x \to 0, \quad (1-x)^\alpha f(x) = O(1) \quad x \to 1.$

Finite Measure Case ($B < 2$)

Invariant density

$$\rho(x) = \frac{2-B}{2}\{x^{1-B} + (1-x)^{1-B}\}.$$

$\{\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n\}$

Birkhoff's ergodic theorem tells us that the probability density function of the sequence $\{Tx, T^2x, \cdots, T^nx\}$ obeys the invariant density as $n \to \infty$. The distribution of $\mathbf{Y} = f(\mathbf{X})$ is given by

$$\Pr(\mathbf{Y} < x) = \Pr(\mathbf{X} > x^{-1/\alpha}) = 1 - \frac{1}{2}x^{-\frac{2-B}{\alpha}}(1 - x^{-\frac{B-1}{\alpha}}).$$

Therefore

$$1 - \Pr(\mathbf{Y} < x) \sim \frac{1}{2}x^{-\frac{2-B}{\alpha}} \quad x \to \infty.$$

Infinite Measure Case ($B \geq 2$)

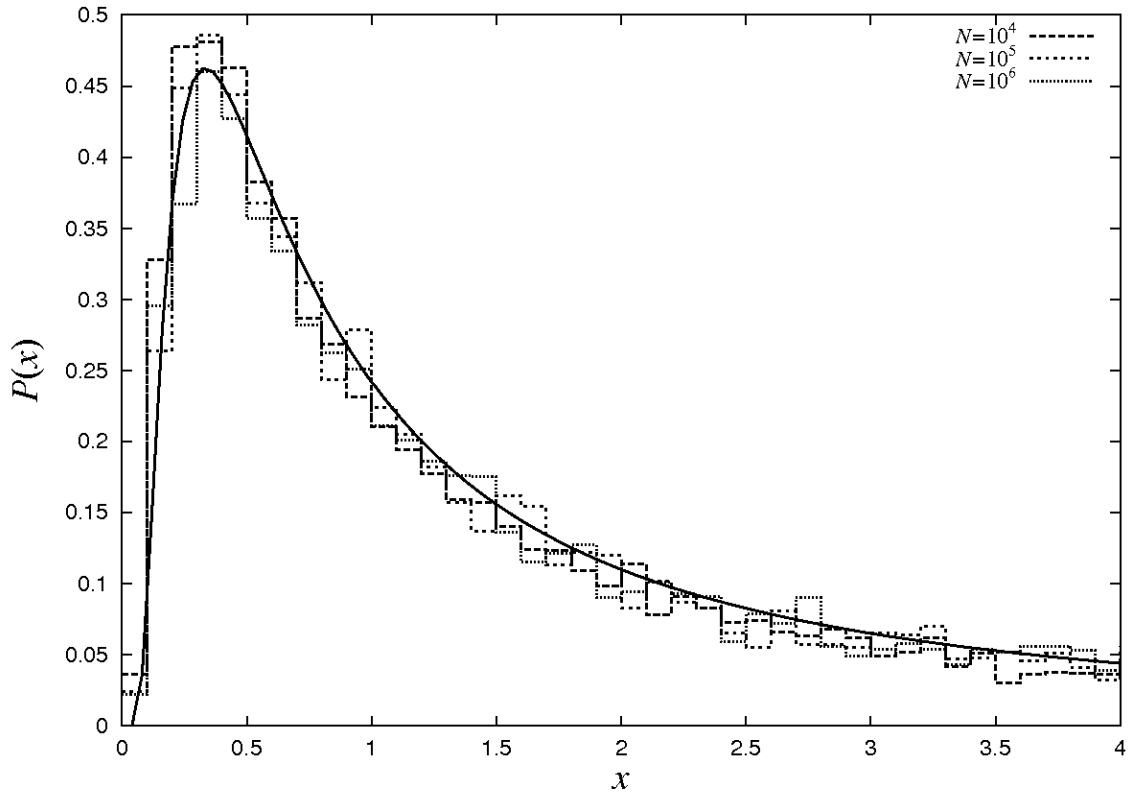


Fig. 1. The probability density function for the scaled time average of $f(x)$ ($B = 3.0, \beta = 2.0$). The fitting curve is a stable distribution with $\gamma = 2.0$.

Convergence to the invariant density

THEOREM Let $T : [0,1] \to [0,1]$ satisfy the conditions (i)-(iv) with return index α. Then, for all Riemann-integrable functions u on $[0,1]$,

$$w_n(T)P^n u \to \left(\frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\int_0^1 u d\lambda\right) h$$

uniformly on compact subsets of $(0,1]$.

$$\boxed{w_n \sim \begin{cases} \log n, & \alpha = 1 \\ n^{1-\alpha}, & \alpha < 1 \end{cases}}$$

Lasota-Yorke map

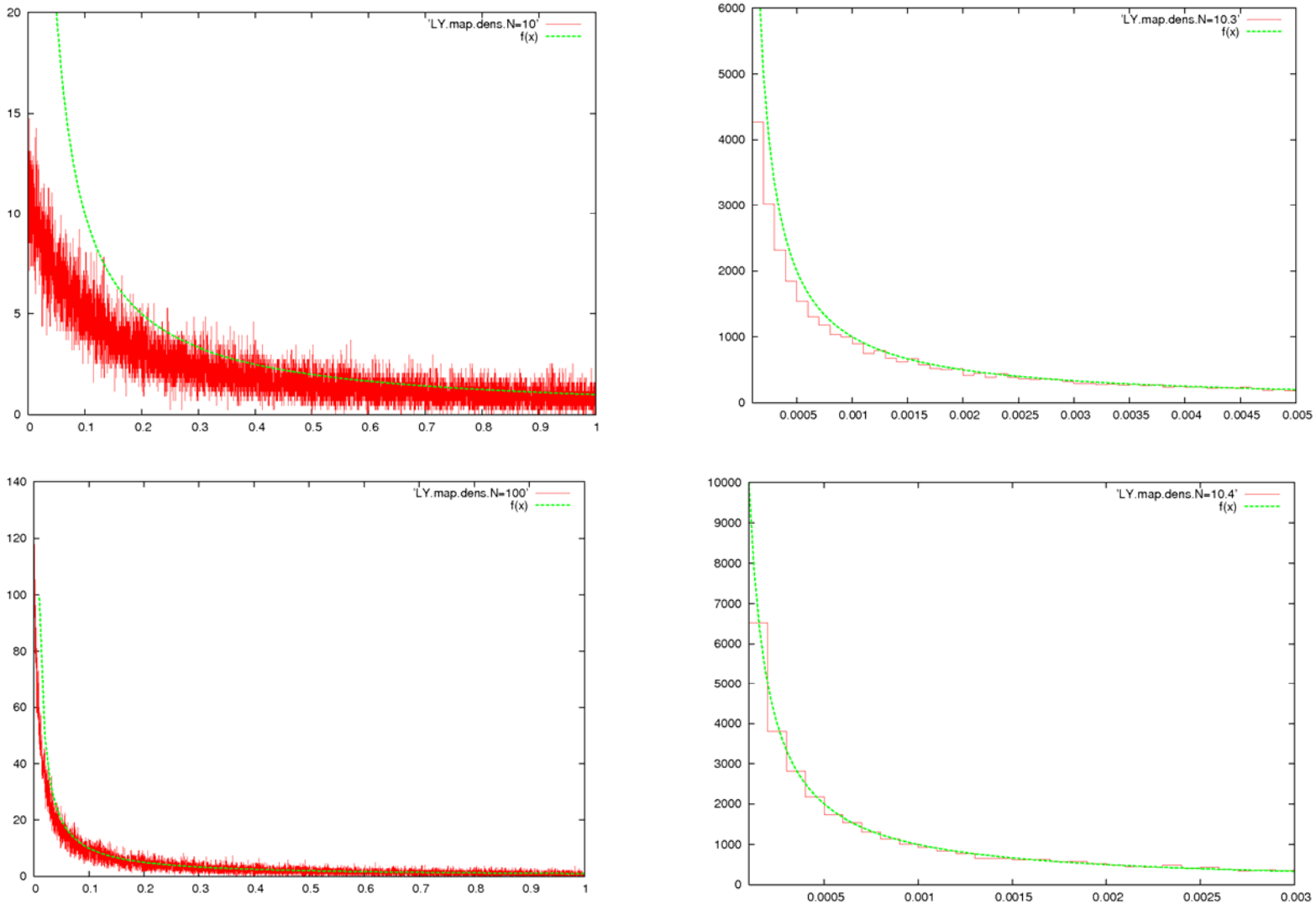
$$T(x) = \begin{cases} x/(1-x), & x \in [0,1/2), \\ 2x-1, & x \in [1/2,1], \end{cases}$$

which has invariant density $h(x) = 1/x$.

$$(\log n)P^n u \to \left(\int_0^1 u d\lambda\right) h \quad n \to \infty$$

Numerical Simulations

$(\log n) P^n_u(x)$



x x

The Scaling Exponent

Assumption $P^n u \to \frac{1}{2}\delta(x) + \frac{1}{2}\delta(1-x)$

Let P be the Perron-Frobenius operator. (Evolution of the density) For $u \in L^1(0,1)$ and $\int_0^1 u(x)dx = 1$

$$P^n u = \begin{cases} 1/2\epsilon_n & (x < \epsilon_n) \\ 0 & (\epsilon \leq x \leq 1-\epsilon_n) \\ 1/2\epsilon_n & (1-\epsilon_n < x) \end{cases} \quad \text{and } \epsilon \propto n^{-\frac{1}{B-1}}.$$

We have $\langle f \circ T^n \rangle = \int_0^1 f(x) P^n u(x) dx = O(\epsilon_n^{-\beta}) = O(n^{\frac{\beta}{B-1}})$.

Therefore $\sum_{k=0}^{n-1} \langle f \circ T^n \rangle \propto n^{\frac{\beta}{B-1}+1}$

Concluding Remarks

In infinite measure dynamical systems the time average of some observation functions converges in distribution (**Generalized Arcsine Law, Stable Law**).

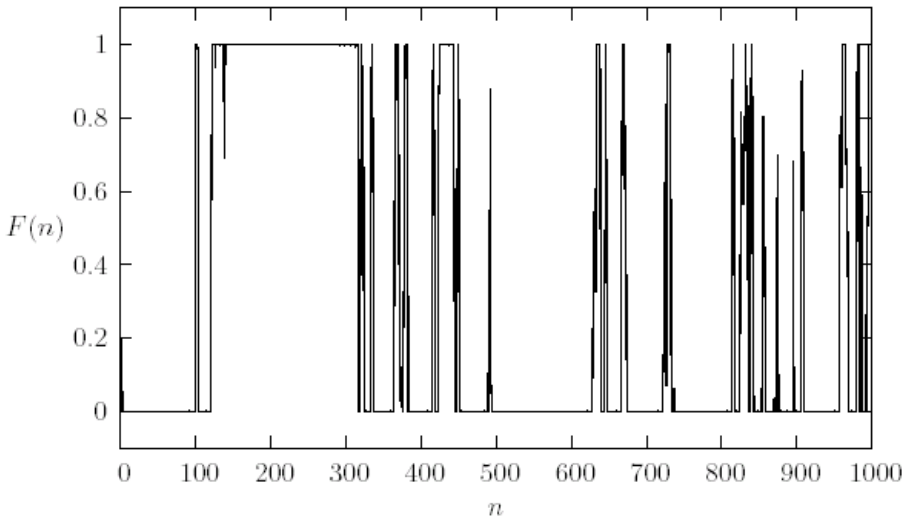
$f(x)$	invariant measure	γ	distribution of $\mathbf{X}$
L^1_m	**finite**	1	*delta* (Birkhoff, 1931)
L^1_m	**infinite**	$\frac{1}{B-1}$	*Mittag-Leffler* (Aaronson, 1981)
non-L^1_m	**finite**	$\frac{\alpha}{2-B}$	*Stable* (**Akimoto [1]**)
$L^1_{\mathrm{loc}}(m,(0,1))$	**infinite**	1	*Generalized Arcsine* (**Akimoto [1]**)
non-L^1_m	**infinite**	$\frac{\alpha}{B-1}+1$	*Stable* (**Akimoto [1]**)

Non-stationary time series (Fluorescence of nanocrystals) → Stationary random variables

Time average

Concluding Remarks

Ergodicity of non-equilibrium state in dynamical system is related to infinite measure systems.



Macroscopic observable $F(n)$ is random.

$$\left(F(n) = \frac{1}{N} \sum_{k=n-N+1}^{n} f(T^k x) \right)$$