第三回 九州大学 産業技術数理研究センター ワークショップ
［兼 第三回 連成シミュレーションフォーラム］
自然現象における階層構造と数理的アプローチ」
Hierarchical Structures in Nature: how we can approach them in mathematics
March 6-8, 2008 (Mar. 8)
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九州大学 情報基盤研究開発センター3階 多目的講習室

渦流の3次元不安定性とそれよって誘導されるドリフト流：ラグランジュ的アプローチ

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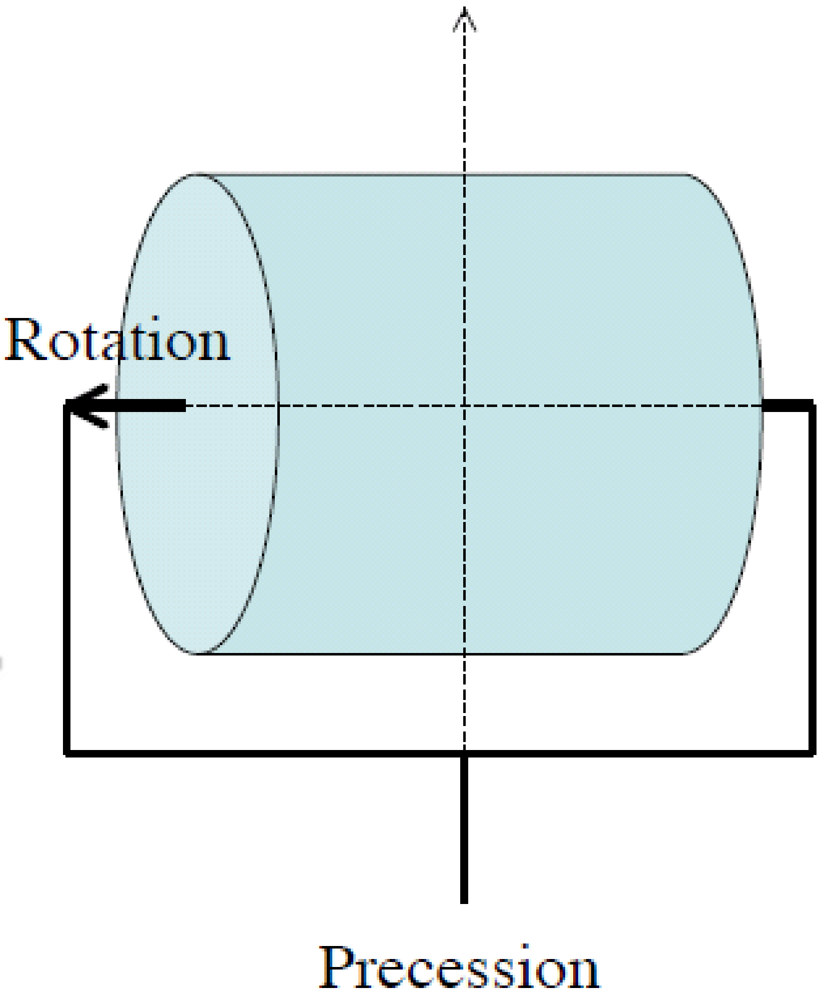
Flows driven by a precessing container

J. Léorat '04

Motivation:

experimental fluid dynamos

What is the maximal speed which may be driven in a closed container of given size, using a given mechanical power?

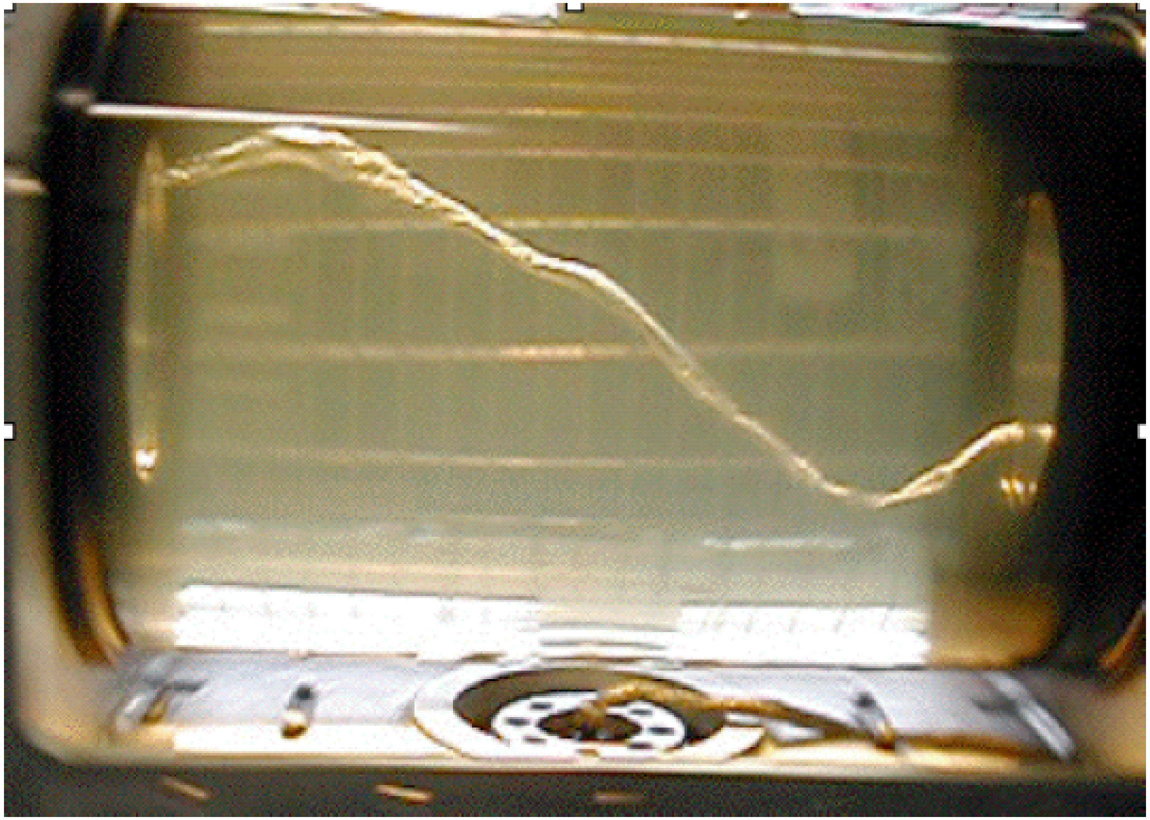


Flows driven by a precessing container

J. Léorat '04

ATER experiment : laminar regime

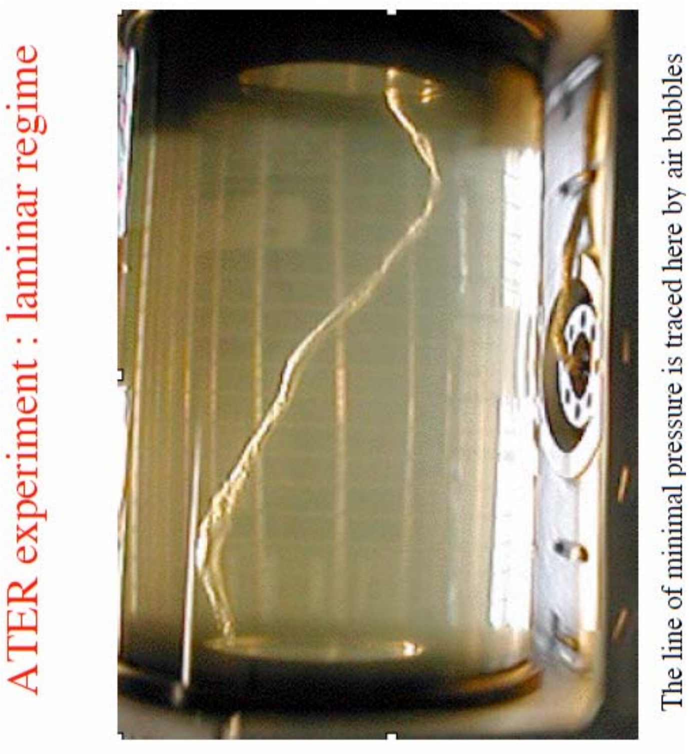
ATER =
Agitateur pour la
Turbulence en
Rotation

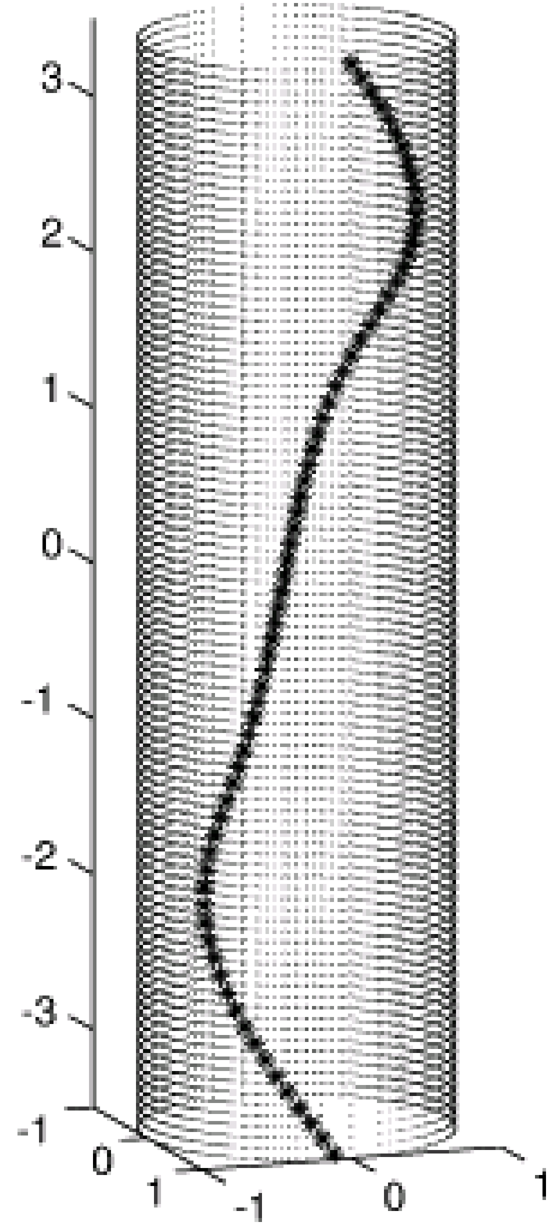


The line of minimal pressure is traced here by air bubbles

A line of local pressure minimum (Theory)

Adachi '04



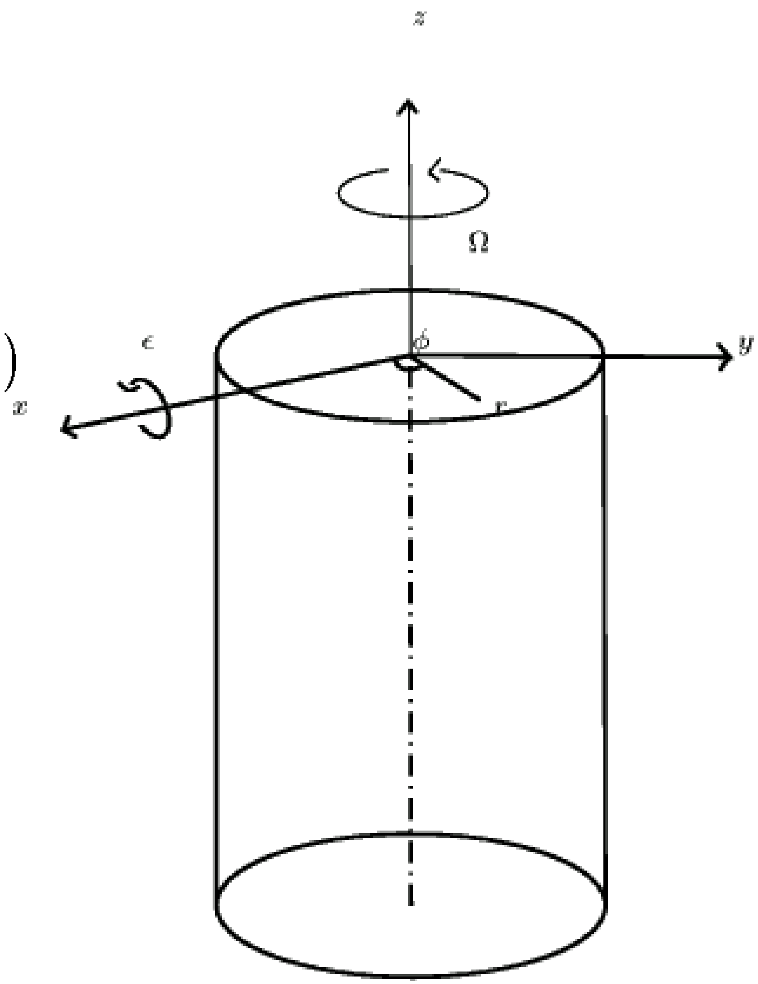


Basic Flow

$$(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} + 2\epsilon\boldsymbol{E}\times\boldsymbol{U} = -\nabla P,$$

$$\nabla\cdot\boldsymbol{U} = 0,$$

$$\text{where}\quad \boldsymbol{E} = \boldsymbol{e}_x = (\cos\phi, \sin\phi, 0)$$



$$U = O(\epsilon^2),\quad V = V_0(r) + O(\epsilon^2),$$

$$W = \epsilon W_1(r,\phi) + O(\epsilon^2)$$

$\boldsymbol{O(\epsilon^0)}$ rigid-body rotation

$$V_0 = r \quad (r \leq 1)$$

$\boldsymbol{O(\epsilon^1)}$ simple shear

$$W_1 = -2r\sin\phi \quad (r \leq 1),$$

$$\rightarrow \quad \boxed{\boldsymbol{U} = (0, r, -2\epsilon r\sin\phi)}.$$

Question: Influence of simple shear upon *Kelvin waves*?

Expand infinitesimal disturbance in ϵ

We seek the disturbance velocity $\tilde{\boldsymbol{u}}$
in a power series of ϵ to first order

$$\tilde{\boldsymbol{u}} = (\boldsymbol{u}_0 + \epsilon \boldsymbol{u}_1 + \cdots) e^{\mathrm{i}(kz - \omega t)} ,$$

with wavenumber k and frequency ω being

$$k = k_0 + \epsilon k_1 + \cdots , \quad \omega = \omega_0 + \epsilon \omega_1 + \cdots .$$

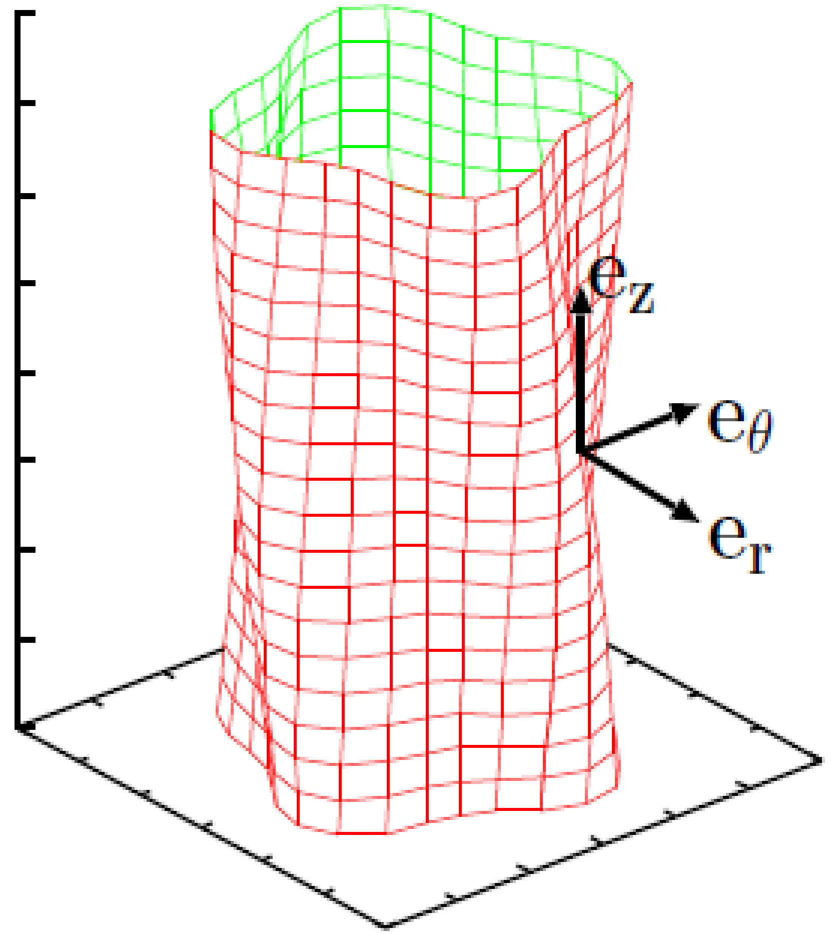
$O(\epsilon^0)$: Kelvin waves

$$\boldsymbol{u}_0 = \boldsymbol{u}_0^{(m)}(r) e^{\mathrm{i} m \phi} , \quad p_0 = \pi_0^{(m)}(r) e^{\mathrm{i} m \phi} .$$

⇨ *the linearized Euler equations*

Example of a Kelvin wave *m=4*

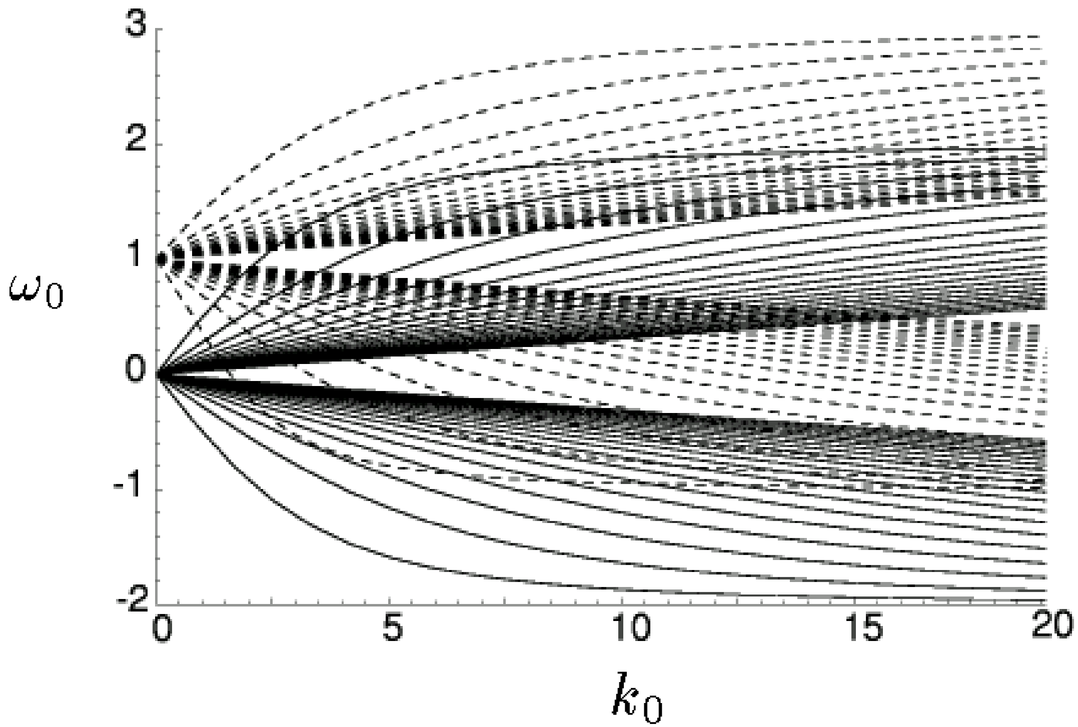
$$\tilde{\boldsymbol{u}} \propto \mathrm{e}^{\mathrm{i}(k_0 z + m\phi - \omega_0 t)}$$



Dispersion relation of Kelvin waves $O(\epsilon^0)$

$$\left(u_0^{(m)}(r=1,\phi)=0 \quad \Rightarrow \right)$$

$$\boxed{J_m(\eta_1)=\frac{\eta_1(\omega_0-m)}{m(\omega_0-m-2)}J_{m+1}(\eta_1); \quad \eta_j := k_0\sqrt{\frac{4}{(\omega_0-m+1-j)^2}-1}}$$



m=0 (solid lines) and *m=1* (dashed lines)

Equations for disturbance of $O(\epsilon^1)$

$$\boldsymbol{u}_1 e^{\mathrm{i}(kz-\omega t)}; \quad \boldsymbol{u}_1 = \{u_1, v_1, w_1, p_1\}$$

$$-\mathrm{i}\omega_0 u_1 + \frac{\partial u_1}{\partial \theta} - 2v_1 + \frac{\partial p_1}{\partial r} = \mathrm{i}\omega_1 u_0 + \mathrm{i}k_0 u_0 \sin\phi + 2w_0 \sin\phi\,,$$

$$\vdots$$

Disturbance field for the *m, m+1* waves

Pose to $O(\epsilon^0)$

$$\boldsymbol{u}_0 = \boldsymbol{u}_0^{(1)} e^{\mathrm{i}m\phi} + \boldsymbol{u}_0^{(2)} e^{\mathrm{i}(m+1)\phi}$$

Then at $O(\epsilon^1)$

$$\Rightarrow \boxed{\boldsymbol{u}_1 = \boldsymbol{u}_1^{(1)} e^{\mathrm{i}m\phi} + \boldsymbol{u}_1^{(2)} e^{\mathrm{i}(m+1)\phi} + \boldsymbol{u}_1^{(3)} e^{\mathrm{i}(m-1)\phi} + \boldsymbol{u}_1^{(4)} e^{\mathrm{i}(m+2)\phi}}$$

Solution of disturbance of $O(\epsilon^1)$

For the *m* wave, we find, from the Euler equations,

$$p_1^{(1)} = C_1^{(1)} J_m(\eta_1 r) - \mathrm{i}k_0 C_0^{(2)} r J_{m+1}(\eta_2 r) - \left\{ \frac{k_1 \eta_1}{k_0} - \frac{4k_0^2 \omega_1}{(\omega_0 - m)^3 \eta_1} \right\} C_0^{(1)} r J_{m+1}(\eta_1 r)$$
$$- \frac{(\omega_0 - m)(\omega_0 - m + 2)(\omega_0 - m - 1)\eta_2}{2k_0(2\omega_0 - 2m - 1)(\omega_0 - m + 1)} i\beta C_0^{(2)} (A_1 + 1) J_m(\eta_2 r),$$

where

$$A_1 = \omega_0^2 - (2m+1)\omega_0 + m^2 + m,$$
$$\eta_1^2 = \left[\frac{4}{(\omega_0 - m)^2} - 1 \right] k_0^2, \quad \eta_2^2 = \left[\frac{4}{(\omega_0 - m - 1)^2} - 1 \right] k_0^2$$

(radial wave numbers)

→ **Disturbance field $\tilde{\boldsymbol{u}}_1$ is explicitely written out!**

Solvability condition and growth rate

The boundary conditions of $O(\epsilon^1): \quad u_1^{(1)} = u_1^{(2)} = 0$

$$\Longrightarrow \quad \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} \begin{bmatrix} C_0^{(1)} \\ C_0^{(2)} \end{bmatrix} = 0$$

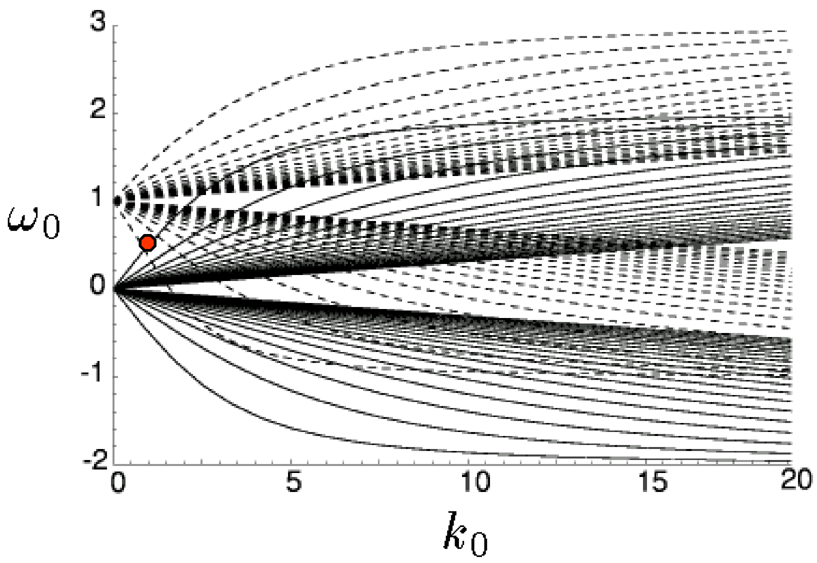
where, for example,

$$\hat{M}_{11} = \mathrm{i}\left[\frac{k_1(k_0^2+m^2)}{k_0(m-\omega_0)} - \frac{2\omega_1(2k_0^2+m(m+\omega_0))}{(m-\omega_0)^2(m-2-\omega_0)(m+2-\omega_0)}\right] J_{m+1}(\eta_2)$$

The solvability condition $\hat{M}_{11}\hat{M}_{22} - \hat{M}_{12}\hat{M}_{21} = 0$ leads to

$$\begin{aligned} {\sigma_{1max}}^2 &= (m-2-\omega_0)(m-1-\omega_0)(m-\omega_0)^2(m+1-\omega_0)^2(m+2-\omega_0) \\ &\quad \times (m+3-\omega_0)A_1(B_2)^2/[16k_0^2(2m+1-2\omega_0)^2 B_1]; \\ B_1 &:= \{2k_0^2+m(m+\omega_0)\}\{2k_0^2+(m+1)(m+1+\omega_0)\}, \\ B_2 &:= k_0^2\{A_1-1\}+m(m+1)\{A_1+1\} \end{aligned}$$

Growth rate of resonance of *(m=0, 1)*



σ_{1max} : growth rate

Δk_1 : unstable band width

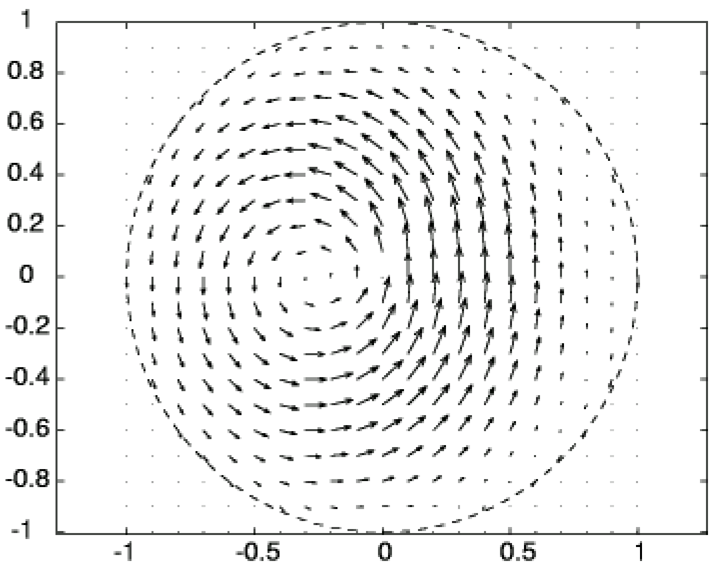
k_0	ω_0	σ_{1max}	Δk_1
*0.877558	0.446491	<u>0.548711</u>	0.751784
1.084229	0.305465	0.104335	0.217218
1.193733	0.233077	0.053563	0.126324
1.262679	0.188693	0.034115	0.086682
*1.220198	0.606867	<u>0.208353</u>	0.486499
1.661231	0.460839	0.495130	1.033362
1.934828	0.373670	0.122560	0.456422
2.125199	0.315028	0.067184	0.291455
*2.040796	0.558633	<u>0.274077</u>	0.864844

Instability occurs at ***every*** intersection points between an upgoing curve of *m=0* and a downgoing curve of *m=1.*

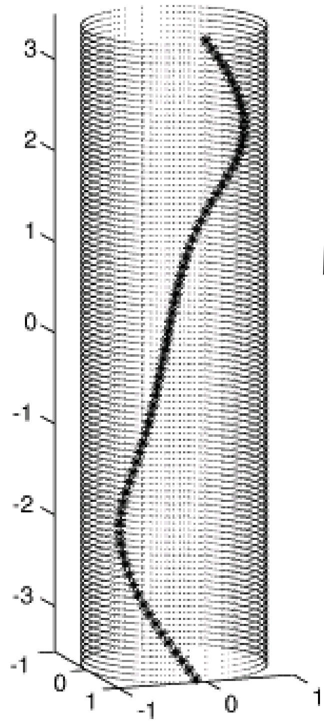
Instability ***NEVER*** occurs at intersection points between upgoing curves and between downgoing curves. **Why?**

Most unstable mode $(k_0, \omega_0) \approx (0.877558, 0.446491)$

The line of local pressure minimum

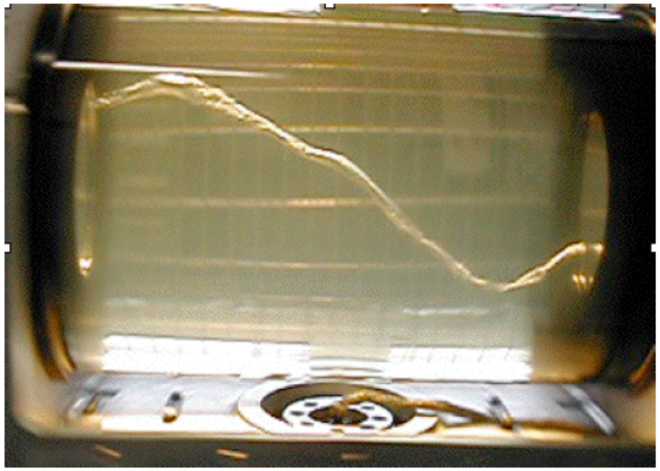


Disturbance velocity field
at the section of *z=0*



Léorat '04

ATER experiment : laminar regime



The line of minimal pressure is traced here by air bubbles

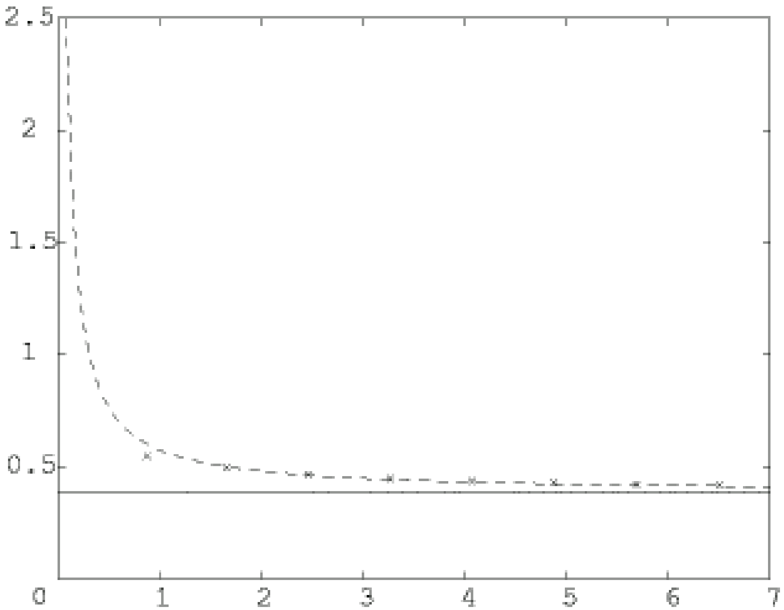
eigen-function

Short-wavelength asymptotics

Large k_0 with *m* fixed, along the principal mode

$$\omega_0 \approx 0.5 \text{ for } (m, m+1) = (0,1)$$

σ_{1max}



k_0

$$\sigma_{1max} = \frac{5\sqrt{15}}{16\pi} + \frac{35(2m+1)}{16\pi^2 k_0} + O\left(\frac{1}{k_0^2}\right),$$

$$\Delta k_1 = \frac{2\sqrt{5}}{\sqrt{3}\pi} k_0 + \frac{4}{\pi^2} + O\left(\frac{1}{k_0}\right)$$

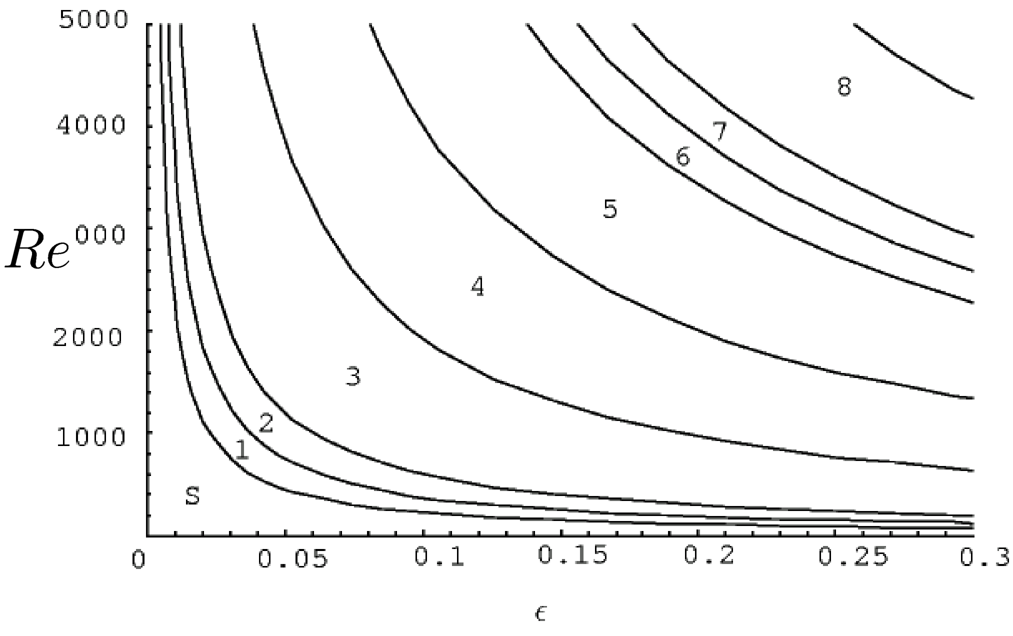
$$\frac{5\sqrt{15}}{16\pi} \approx 0.385253$$

Effect of viscosity

Kelvin wave

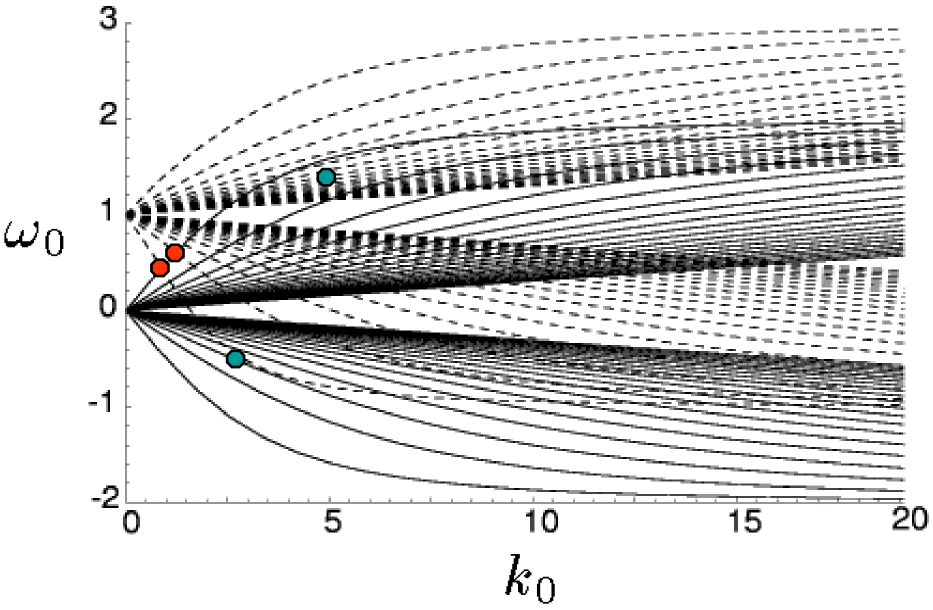
Assume $Re = \dfrac{\Omega R^2}{\nu} \geq O(\epsilon^{-1})$ $\qquad \dfrac{1}{Re}\nabla^2 \tilde{\boldsymbol{u}}_0$

$$\boxed{\epsilon\sigma_{1max}^{\nu} = \left[\epsilon^2\sigma_{1max}^2 - \frac{4k_0^4}{Re^2}\frac{(A_1+1)(3A_1+1)}{(m-\omega_0)^4(m+1-\omega_0)^4}\right]^{1/2} - \frac{2k_0^2}{Re}\frac{2A_1+1}{(m-\omega_0)^2(m+1-\omega_0)^2}}$$



The most unstable mode (m-1,m) for given (ϵ, Re)

Why stable and why unstable?

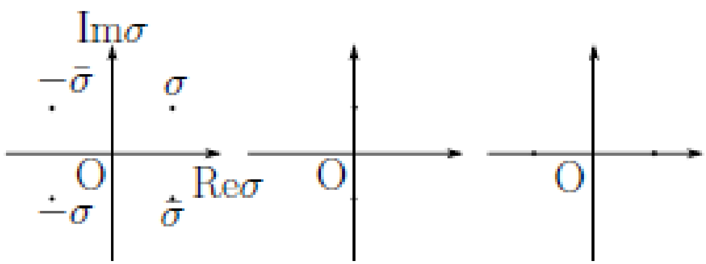


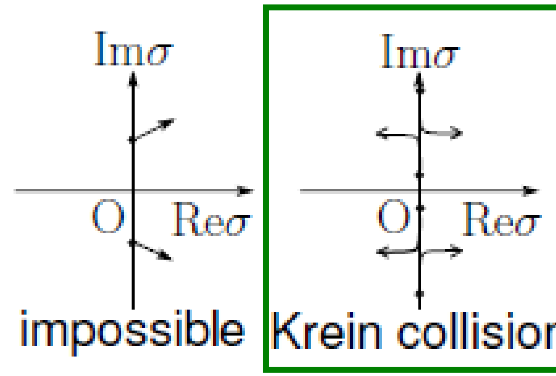
Instability ***NEVER*** occurs at intersection points between upgoing curves and between downgoing curves. **Why?**

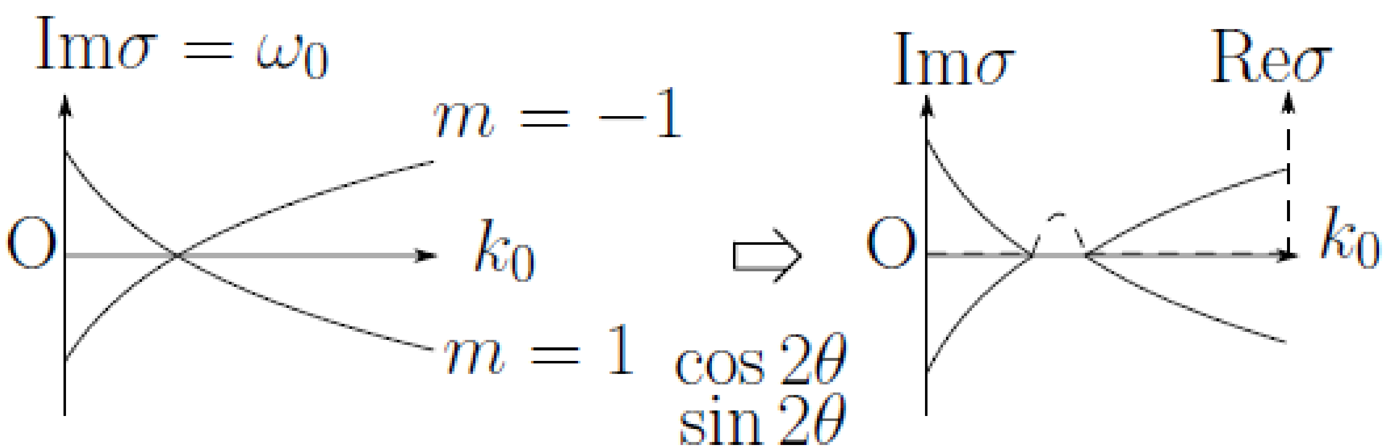
Krein's theory of Hamiltonian spectra

Spectra of a *finte*-dimensional Hamilton system

$$\boldsymbol{z}(t) \propto \mathrm{e}^{\sigma t}$$







Energy of a Kelvin wave

$$\boldsymbol{u} = \boldsymbol{U} + \tilde{\boldsymbol{u}}; \quad \boldsymbol{U} = V_0 \boldsymbol{e}_\theta$$

base flow

disturbance

(averaged)
Excess energy
for generating
a Kelvin wave

$$\frac{1}{2}\int \overline{\boldsymbol{u}^2}\mathrm{d}V - \frac{1}{2}\int \boldsymbol{U}^2\mathrm{d}V$$
$$= \int \boldsymbol{U}\cdot\overline{\tilde{\boldsymbol{u}}}\mathrm{d}V + \frac{1}{2}\int \overline{\tilde{\boldsymbol{u}}^2}\mathrm{d}V$$

$$O(\varepsilon^0)$$

(no strain)

$$\tilde{\boldsymbol{u}} = \alpha\tilde{\boldsymbol{u}}_{01} + \alpha^2\boxed{\tilde{\boldsymbol{u}}_{02}}$$

Kelvin wave

stationary component ???

Carins' formula (Carins '79)

Boundary $\eta(\theta, z, t) = 1 + A_0^{(m)} \cos(m\theta + k_0 z - \omega_0 t)\,.$

Boundary pressure $p_< = p|_{r=\eta-}$, $p_> = p|_{r=\eta+}$;

$$p_> = D_>(k_0, \omega_0) A_0^{(m)} \cos(m\theta + k_0 z - \omega_0 t)\,, \quad p_< = D_<(k_0, \omega_0) A_0^{(m)} \cos(m\theta + k_0 z - \omega_0 t)\,.$$

$\Longrightarrow$ dispersion relation : $D(k_0, \omega_0) := D_>(k_0, \omega_0) - D_<(k_0, \omega_0) = 0$

Fukumoto '03

$$E = -\frac{1}{2}\pi\omega_0 \frac{\partial D}{\partial \omega_0} A^2$$

Where is the boundary?

Difficulty in Eulerian treatment

$$\boldsymbol{u} = \underset{\text{base flow}}{\boldsymbol{U}} + \underset{\text{disturbance}}{\tilde{\boldsymbol{u}}}; \qquad \tilde{\boldsymbol{u}} = \alpha \tilde{\boldsymbol{u}}_{01} + \alpha^2 \tilde{\boldsymbol{u}}_{02}$$

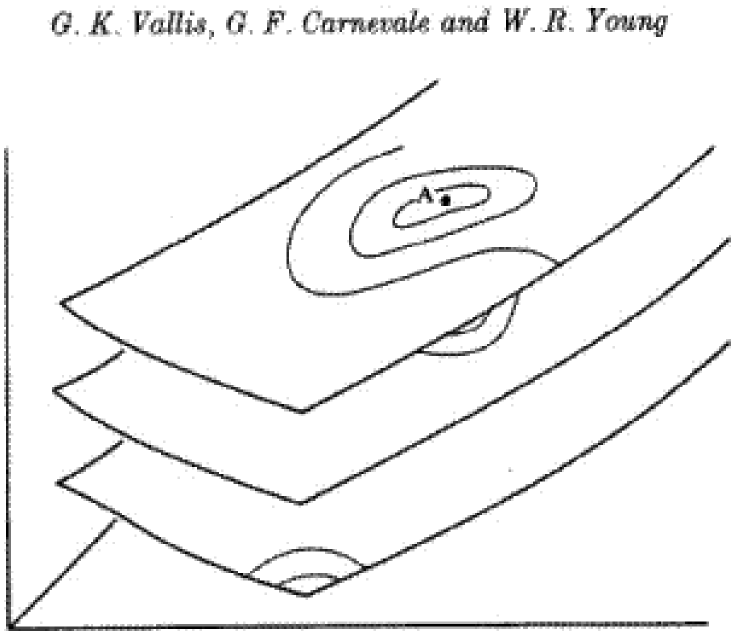
Excess energy

$$\frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}V - \frac{1}{2}\int \boldsymbol{U}^2 \mathrm{d}V$$
$$= \alpha \delta H + \alpha^2 \delta^2 H;$$

$$\delta H = \int \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{01} \mathrm{d}V, \quad \delta^2 H = \frac{1}{2}\int \left(\tilde{\boldsymbol{u}}_{01}^2 + 2\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{02}\right) \mathrm{d}V$$

- * $\delta H \neq \text{const.} \quad \delta^2 H \neq \text{const.}$
- * Complicated calculation would be required for $\tilde{\boldsymbol{u}}_{02}$

Steady Euler flows



iso-vortical sheets

Kinematically accessible variation
(= preservation of circulation)

$$\omega := \frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x},t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} \quad \Rightarrow \quad \omega = \tilde{\omega};$$

$$\frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x},t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$
$$= \frac{1}{2}\epsilon_{pqr}\tilde{\omega}_r(\tilde{\boldsymbol{x}},t)\mathrm{d}\tilde{x}_p \wedge \mathrm{d}\tilde{x}_q$$
$$(\tilde{\omega}_r = \omega_r + \delta\omega_r)$$

Theorem (Kelvin, Arnold '66) A steady Euler flow is a coditional extremum of energy *H* on an iso-vortical sheet (= w.r.t. kinematically accessible variations).

Variational principle for *stationary* vortical region

☆ Volume preserving displacement of fluid particles:

$$\boldsymbol{x} \to \boldsymbol{x} + \delta\boldsymbol{\xi}(\boldsymbol{x})\,; \qquad \nabla\cdot\delta\boldsymbol{\xi} = 0$$

☆ Iso-vorticity: $\boldsymbol{\omega}(\boldsymbol{x}) \to \boldsymbol{\omega}(\boldsymbol{x}) + \delta\boldsymbol{\omega}(\boldsymbol{x})\,;\;\; \delta\boldsymbol{\omega} = \nabla\times(\delta\boldsymbol{\xi}\times\boldsymbol{\omega})$

$$\int_{\tilde{S}}\{\boldsymbol{\omega}+\delta\boldsymbol{\omega}\}\cdot\boldsymbol{n}\mathrm{d}A = \int_S \boldsymbol{\omega}\cdot\boldsymbol{n}\mathrm{d}A \quad (S\text{: material surface})$$

Then. using $\boldsymbol{A}\cdot\delta\boldsymbol{\omega} = -\nabla\cdot(h\delta\boldsymbol{\xi} + \boldsymbol{A}\times(\delta\boldsymbol{\xi}\times\boldsymbol{\omega}))\,,$

$$\delta H = \rho\int \boldsymbol{u}\cdot\delta\boldsymbol{u}\mathrm{d}V = \rho\int \boldsymbol{A}\cdot\delta\boldsymbol{\omega}\mathrm{d}V$$

$$= -\rho\int\{h\delta\boldsymbol{\xi} + \boldsymbol{A}\times(\delta\boldsymbol{\xi}\times\boldsymbol{\omega})\}\cdot\boldsymbol{n}\,\mathrm{d}A = 0$$

$$(\boldsymbol{u} = \nabla\times\boldsymbol{A})$$

First and second variations

The first variation

($\mathcal{P}$: projection operator)

Given $\delta\boldsymbol{\xi}$, which satisfies

$$\boxed{\frac{\partial}{\partial t}\delta\boldsymbol{\xi} = \nabla \times (\boldsymbol{U} \times \delta\boldsymbol{\xi}) + \delta\boldsymbol{u}\,; \quad \delta\boldsymbol{u} := \mathcal{P}\,(\delta\boldsymbol{\xi} \times \boldsymbol{\omega})\ , \quad (\mathcal{P}:\ \nabla \cdot \delta\boldsymbol{u} = 0)}$$

Then $\delta\boldsymbol{\omega} := \nabla \times \delta\boldsymbol{u} = \nabla \times (\delta\boldsymbol{\xi} \times \boldsymbol{\omega})$ is a solution of

$$\frac{\partial}{\partial t}\delta\boldsymbol{\omega} = \nabla \times (\boldsymbol{U} \times \delta\boldsymbol{\omega}) + \nabla \times (\delta\boldsymbol{u} \times \boldsymbol{\omega})$$

The second variation

Further, given $\delta\boldsymbol{\eta}$, which satisfies

$$\boxed{\frac{\partial}{\partial t}\delta\boldsymbol{\eta} = \nabla \times (\boldsymbol{U} \times \delta\boldsymbol{\eta}) + \mathcal{P}\,(\delta\boldsymbol{\eta} \times \boldsymbol{\omega}) + \nabla \times (\delta\boldsymbol{\xi} \times \delta\boldsymbol{u}) - \mathcal{P}\,(\delta\boldsymbol{\xi} \times \delta\boldsymbol{u})}$$

Then $\delta^2\boldsymbol{\omega} := \frac{1}{2}\left[\nabla \times (\delta\boldsymbol{\xi} \times \delta\boldsymbol{\omega}) - \nabla \times (\delta\boldsymbol{\eta} \times \boldsymbol{\omega})\right]$ is a solution of

$$\frac{\partial}{\partial t}\delta^2\boldsymbol{\omega} = \nabla \times \left(\boldsymbol{U} \times \delta^2\boldsymbol{\omega}\right) + 2\nabla \times (\delta\boldsymbol{u} \times \delta\boldsymbol{\omega}) + \nabla \times \left(\boldsymbol{U} \times \delta^2\boldsymbol{\omega}\right)$$

Wave energy in terms of iso-vortical disturbance

$$\boldsymbol{u} = \boldsymbol{U} + \delta\boldsymbol{u} + \delta^2\boldsymbol{u}$$

Excess energy

$$\Delta H := \frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}V - \frac{1}{2}\int \boldsymbol{U}^2 \mathrm{d}V$$
$$= \cancel{\delta H} + \delta^2 H;$$

$$\delta H = \int \boldsymbol{U}\cdot\delta\boldsymbol{u}\,\mathrm{d}V = 0$$ by Arnold's theorem

$$\delta^2 H = \frac{1}{2}\int \left(\delta\boldsymbol{u}\cdot\delta\boldsymbol{u} + 2\boldsymbol{U}\cdot\delta^2\boldsymbol{u}\right)\mathrm{d}V$$

It is proved that $\dfrac{\mathrm{d}}{\mathrm{d}t}\delta^2 H = 0$ ⟹ $\boxed{\delta^2 H \text{ is the wave-energy}}$

and that $\delta\boldsymbol{\eta}$ does not contribute to $\delta^2 H$

⟹ $\delta^2\boldsymbol{u} \approx \mathcal{P}\left(\delta\boldsymbol{\xi}\times\delta\boldsymbol{\omega}\right)$; $\delta\boldsymbol{\xi},\ \delta\boldsymbol{\omega}$ are ***linear*** disturbances!!

Energy formula

Euler equations

$$\frac{\partial \boldsymbol{u}}{\partial t} = B(\boldsymbol{u}, \boldsymbol{u}); \quad B(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{P}[\boldsymbol{w} \times (\nabla \times \boldsymbol{v})]$$

Kinetic energy

$$\delta^2 H = \frac{1}{2}\langle B(\boldsymbol{U}, \delta\boldsymbol{\xi}), B(\boldsymbol{U}, \delta\boldsymbol{\xi})\rangle + \frac{1}{2}\langle [\boldsymbol{U}, \delta\boldsymbol{\xi}], B(\boldsymbol{U}, \delta\boldsymbol{\xi})\rangle$$

$$\Rightarrow \boxed{\delta^2 H = \frac{1}{2}\left\langle \frac{\partial \delta\boldsymbol{\xi}}{\partial t} \times \delta\boldsymbol{\xi}, \boldsymbol{\Omega} \right\rangle}.$$

Energy of Kelvin waves

Lagrangian dispalcement $\delta\boldsymbol{\xi} = \mathrm{Re}\left[C_0\hat{\boldsymbol{\xi}}(r;\omega_0,m,k_0)e^{\mathrm{i}(m\theta+k_0-\omega_0 t)}\right];$

$$\hat{\xi}_r^{(m)} = \frac{\omega_0 - m}{\sqrt{4-(\omega_0-m)^2}}\left\{\frac{m}{r}(\omega_0-m-2)J_m(\eta_m r) - (\omega_0-m)\eta_m J_{m+1}(\eta_m r)\right\},$$

$$\hat{\xi}_\theta^{(m)} = \mathrm{i}\frac{\omega_0 - m}{\sqrt{4-(\omega_0-m)^2}}\left\{-\frac{m}{r}(\omega_0-m-2)J_m(\eta_m r) - 2\eta_m J_{m+1}(\eta_m r)\right\},$$

$$\hat{\xi}_z^{(m)} = -\mathrm{i}k_0\sqrt{4-(\omega_0-m)^2}J_m(\eta_m r), \qquad \text{where} \quad \eta_m := k_0\sqrt{4/(\omega_0-m)^2-1}.$$

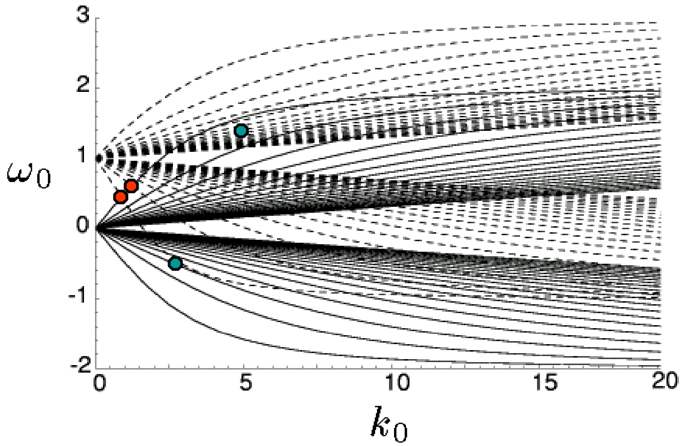
The wave energy per unit length in *z* is $E_0 = \omega_0\mu_0;$

$$\begin{aligned}\mu_0 &= 2\pi\left|C_0\right|^2\frac{\omega_0-m}{2}\int_0^1|\hat{\boldsymbol{\xi}}|^2\mathrm{d}r\\ &= \pi\left|C_0\right|^2\frac{\partial D}{\partial\omega_0}(\omega_0;m,k);\end{aligned}$$

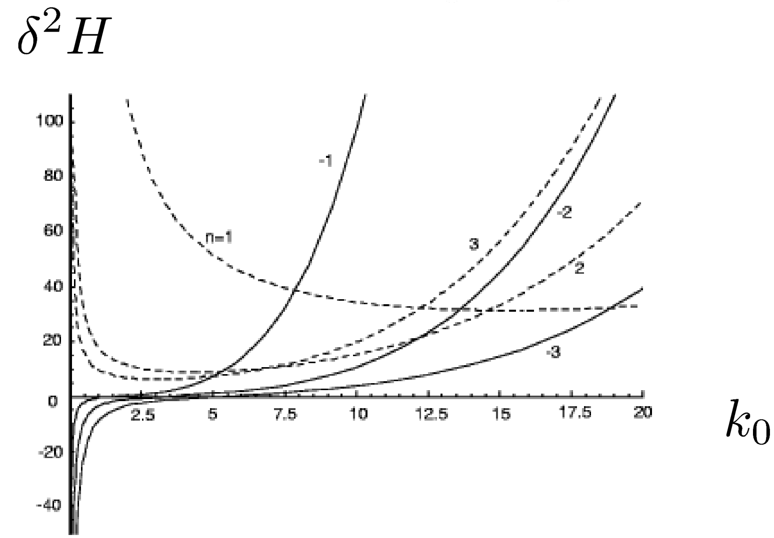
$$D(\omega_0,m,k) := (\omega_0-m)^3 J_m(\eta_m)\left[(\omega_0-m)\eta_m J_{m-1}(\eta_m) - m(\omega_0-m+2)J_m(\eta_m)\right]$$

$\mu_0 = E_0/\omega_0:$ wave action, $D = 0:$ dispersion relation

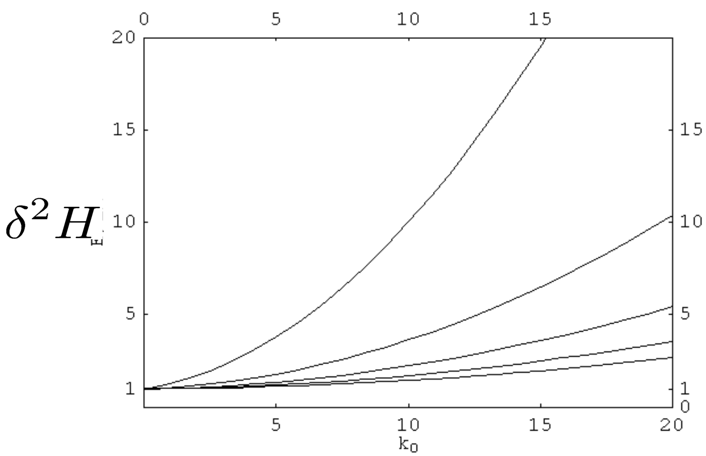
Energy of Kelvin waves



Helical wave *(m=1)*



Buldge wave *(m=0)*



$$E_0^{(m)} = 2\pi\omega_0 \, |C_0|^2 \, \frac{\omega_0 - m}{2} \int_0^1 |\hat{\boldsymbol{\xi}}|^2 \mathrm{d}r$$

The sign of **wave action** $\mu_0 = E_0/\omega_0$ is essential !

Derivation of Energy formula

Let $\Xi(r,\Omega;m,k_0)$, $\Omega\in\mathbb{C}$, be the Laplace transform of $C_0 r\hat{\xi}_r(r;\omega_0,m,k_0)e^{-i\omega_0 t}$,

Laplace transform

$$\Xi(r,\Omega) = \int_0^\infty [C_0 r\hat{\xi}_r(r;\omega_0)e^{-i\omega_0 t}]e^{i\Omega t}dt, \quad \mathrm{Im}(\Omega)>0$$
$$= \frac{iC_0}{\Omega-\omega_0} r\hat{\xi}_r(r;\omega_0).$$

ts inverse transform is represented by

$$C_0 r\hat{\xi}_r(r;\omega_0)e^{-i\omega_0 t} = -\frac{1}{2\pi}\oint_{\Gamma(\omega_0)} \Xi(r,\Omega)e^{-i\Omega t}d\Omega,$$

dispersion relation

$$\mathcal{D}(\Omega) := 2\pi L\int_0^1 \overline{\Xi(r,\overline{\Omega})}\mathcal{E}(\Omega)\Xi(r,\Omega)dr,$$

n would be obtained by the residue of $\mathcal{I}$

$$\mathcal{D}(\Omega) \simeq 2\pi L\frac{|C_0|^2}{(\Omega-\omega_0)^2}D(\Omega;m,k)$$

Action

$$\mu_o = E_0/\omega_0$$

$$2\mu_0 = \frac{1}{2\pi i}\oint_{\Gamma(\omega_j)} \mathcal{D}(\Omega)d\Omega.$$

$$2\mu_0 = \frac{1}{2\pi i}\oint \mathcal{D}(\Omega)d\Omega,$$
$$= 2\pi L|C_0|^2\frac{\partial D}{\partial\Omega}(\omega_0;m,k).$$

$$2\mu_0 = 2\pi L|C_0|^2\int_0^1 r\hat{\xi}_r\frac{\partial\mathcal{E}}{\partial\Omega}(\omega_0)r\hat{\xi}_r dr,$$

Drift current

Take the average over a long time $\Rightarrow$ $\overline{\boldsymbol{u}} = \boldsymbol{U} + \overline{\delta^2 \boldsymbol{u}} + O(\alpha^3)$

$$\boxed{\overline{\delta^2 \boldsymbol{u}} = \frac{1}{2}\mathcal{P}(\overline{\delta\boldsymbol{\xi} \times [\nabla \times (\delta\boldsymbol{\xi} \times \boldsymbol{\omega})]} - \overline{\delta\boldsymbol{\eta}} \times \boldsymbol{\omega})}$$

For the ***Rankine vortex***

$$\boldsymbol{U} = V_0 \boldsymbol{e}_\theta\,; \quad V_0 = \begin{cases} r & (r \leq 1) \\ 1/r & (r > 1)\,, \end{cases}$$

Substitute the ***Kelvin wave***

$$\delta\boldsymbol{\xi} = \mathrm{Re}\left[\hat{\boldsymbol{\xi}} \mathrm{e}^{\mathrm{i}(m\theta + +k_0 z - \omega_0 t)}\right]$$

$\Downarrow$

$$\mathcal{P}(\overline{\delta\boldsymbol{\xi} \times [\nabla \times (\delta\boldsymbol{\xi} \times \boldsymbol{\omega})]} = \begin{cases} \mathrm{i}k_0(0,\ \hat{\xi}_z^* \hat{\xi}_r - \hat{\xi}_r^* \hat{\xi}_z,\ \hat{\xi}_r^* \hat{\xi}_\theta - \hat{\xi}_\theta^* \hat{\xi}_r) & (r \leq 1) \\ 0 & (r > 1)\,. \end{cases}$$

$$I_z := \int \overline{\delta^2 u_z} \mathrm{d}A = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \left(\hat{\xi}_r^* \hat{\xi}_\theta - \hat{\xi}_\theta^* \hat{\xi}_r\right)$$

- There is no contribution from $\overline{\delta\boldsymbol{\eta}}$
- For *2D* wave, $I_z = 0$

genuinly *3D* effect !!

Drift current caused by Kelvin waves

Displacement vector of *m* wave $\delta\boldsymbol{\xi} = \mathrm{Re}\left[C_0\hat{\boldsymbol{\xi}}(r;\omega_0,m,k_0)e^{\mathrm{i}(m\theta+k_0z-\omega_0t)}\right];$

$$\hat{\xi}_r^{(m)} = \frac{\omega_0-m}{\sqrt{4-(\omega_0-m)^2}}\left\{\frac{m}{r}(\omega_0-m-2)J_m(\eta_m r)-(\omega_0-m)\eta_m J_{m+1}(\eta_m r)\right\},$$

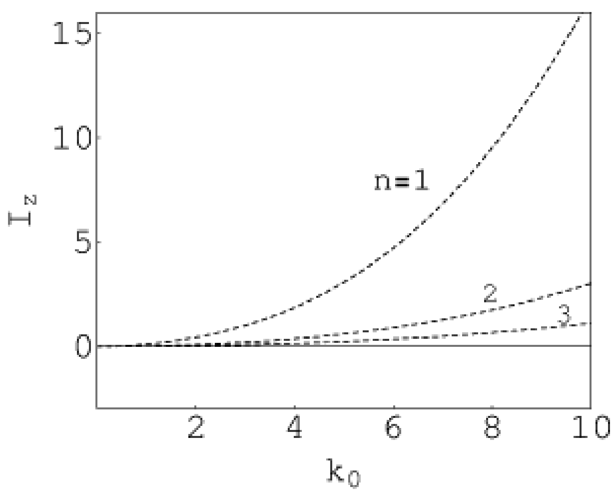
$$\hat{\xi}_\theta^{(m)} = \mathrm{i}\frac{\omega_0-m}{\sqrt{4-(\omega_0-m)^2}}\left\{-\frac{m}{r}(\omega_0-m-2)J_m(\eta_m r)-2\eta_m J_{m+1}(\eta_m r)\right\},$$

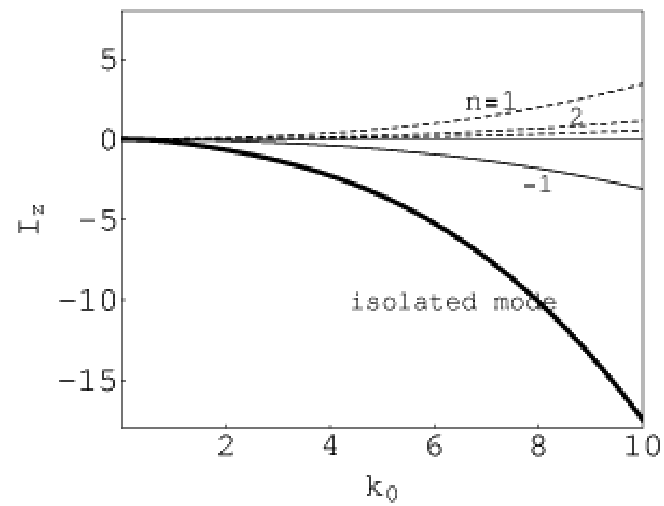
$$\hat{\xi}_z^{(m)} = -\mathrm{i}k_0\sqrt{4-(\omega_0-m)^2}J_m(\eta_m r), \qquad \text{where} \quad \eta_m := k_0\sqrt{4/(\omega_0-m)^2-1}.$$

Flow-flux, of *m* wave, in the *axial* direction

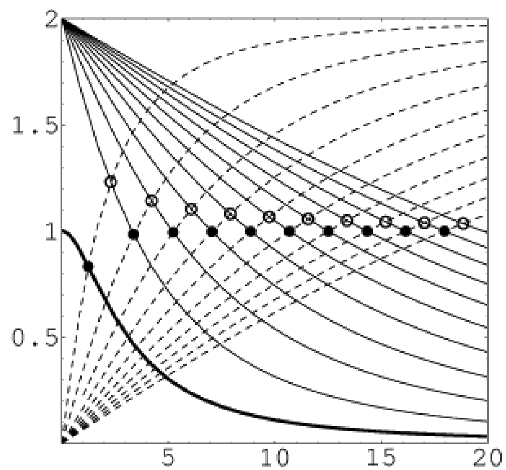
$$\boxed{\begin{aligned} I_z^{(m)}(k_0,\omega_0) &= \int \overline{\delta^2 u_z}\,\mathrm{d}A \\ &= |C_0|^2\frac{k_0}{\eta_m^4}\left\{\frac{2k_0^2}{\omega_0-m}\left[\eta_m^2\left(J_m'(\eta_m)\right)^2+2\eta_m J_m(\eta_m)J_m'(\eta_m)\right.\right. \\ &\qquad \left.\left.+(\eta_m^2-m^2)J_m^2(\eta_m)\right]-m(\eta_m^2+2k_0^2)J_m^2(\eta_m)\right\} \end{aligned}}$$

Axial flow-flux of
buldge wave *(m=0)*, elliptic wave *(m=2)*





Dispersion relation

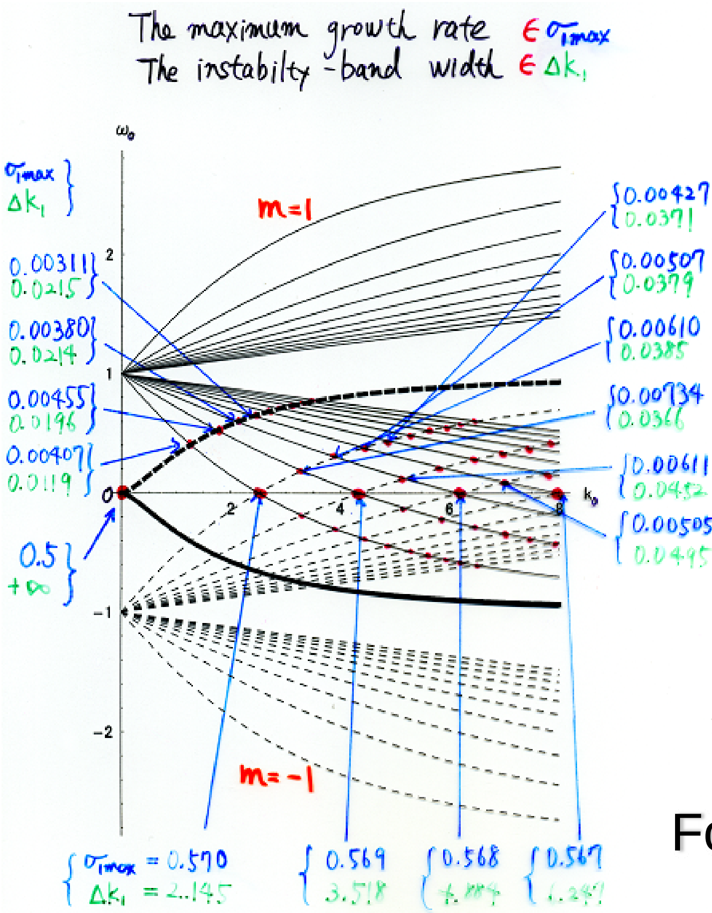


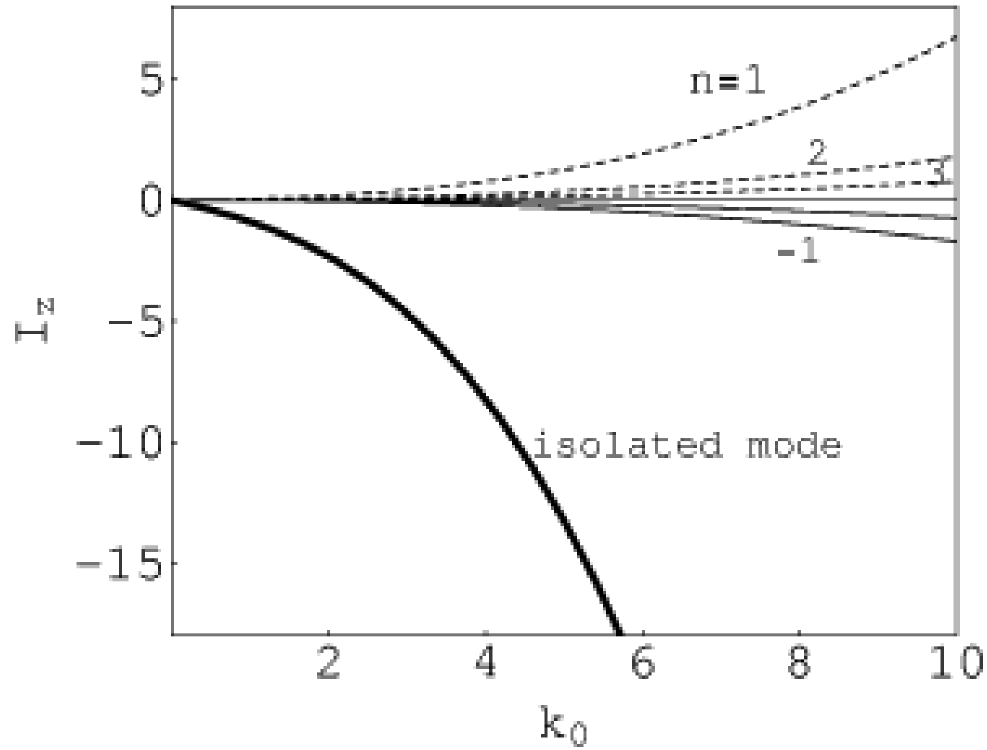
m=0 (dashed lines) and *m=2* (solid lines)

For the *principal* mode,

$k_0 =$	$I_z^{(0)} + I_z^{(2)} =$
• 1.242,	-1.242
• 3.370,	-0.2443
• 7.058,	-0.09046
• 8.882,	-0.06828
• 12.521,	-0.04564

Axial flow-flux of a helical wave *(m=1)*





For the *principal* mode （=stationary)

$$k_0 = \qquad I_z^{(-1)} + I_z^{(1)} = 0$$

- 2.505
- 4.349

Pseudomomentum

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_e(\boldsymbol{x}) + \epsilon\tilde{\boldsymbol{u}}(\boldsymbol{x},t) + \frac{\epsilon^2}{2}\tilde{\tilde{\boldsymbol{u}}}(\boldsymbol{x},t) + O(\epsilon^3), \quad \begin{cases} \tilde{\boldsymbol{u}} = \mathcal{P}[\boldsymbol{\xi}\times\boldsymbol{w}_e], \\ \tilde{\tilde{\boldsymbol{u}}} = \mathcal{P}[\boldsymbol{\xi}\times(\nabla\times(\boldsymbol{\xi}\times\boldsymbol{w}_e)) - \boldsymbol{\eta}\times\boldsymbol{w}_e], \end{cases}$$

$\mathcal{P}$: projection operator that maps any vector field into solenoidal one.

Let ***v*** be an arbitrary vector field.

Note that, if $\boldsymbol{v}$ satisfies $\mathcal{P}[\boldsymbol{v}\times\boldsymbol{w}_e] = 0$ (namely, if $\mathcal{L}_{\boldsymbol{v}}\boldsymbol{w}_e = 0$), this expression reduces to

$$\int_V \boldsymbol{u}\cdot\boldsymbol{v}d^3x = \int_V \boldsymbol{u}_e\cdot\boldsymbol{v}d^3x + \frac{\epsilon^2}{2}\int_V \boldsymbol{w}_e\cdot(\boldsymbol{\xi}\times\mathcal{L}_{\boldsymbol{v}}\boldsymbol{\xi})d^3x + O(\epsilon^3), \tag{101}$$

and $\boldsymbol{\eta}$ is not required for this computation. $\quad \mathcal{L}_{\boldsymbol{v}}\boldsymbol{\xi} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}$

For a Kelvin wave, $\quad \boldsymbol{\xi} = \mathrm{Re}\left[C_0\hat{\boldsymbol{\xi}}(r;\omega_0,m,k_0)e^{\mathrm{i}(m\theta+k_0z-\omega_0t)}\right]$ on $\boldsymbol{w}_e = (0,0,2)$

we may choose $\quad \boldsymbol{v} = \boldsymbol{e}_z = (0,0,1)$

$$P_z = \int_V \boldsymbol{u}\cdot\boldsymbol{e}_z d^3x = \frac{\epsilon^2}{2}\tilde{\tilde{P}}_z + O(\epsilon^3),$$

$$\begin{aligned}\tilde{\tilde{P}}_z = \int_V \tilde{\tilde{\boldsymbol{u}}}\cdot\boldsymbol{e}_z d^3x &= \int_V \boldsymbol{w}_e\cdot(\boldsymbol{\xi}\times\partial_z\boldsymbol{\xi})d^3x, \\ &= 2\pi L\frac{|C_0|^2}{2}ik\int_0^1 \boldsymbol{w}_e\cdot\left(\bar{\hat{\boldsymbol{\xi}}}\times\hat{\boldsymbol{\xi}}\right)rdr, \\ &= k\mu_0.\end{aligned}$$

pseudomomentum

Lagrangian description of wave mean-flow interaction

Andrews & McIntyre '78

$$\begin{cases} \boldsymbol{x} : & \text{mean position} \\ \boldsymbol{\Xi}(\boldsymbol{x},t) := \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x},t) : & \text{displaced position} \end{cases}$$

Assume $\overline{\boldsymbol{\xi}(\boldsymbol{x},t)} = \mathbf{0}$

Lagrangian mean operator

$$\overline{\varphi(\boldsymbol{x},t)}^{\mathrm{L}} := \overline{\varphi^{\xi}(\boldsymbol{x},t)}; \quad \varphi^{\xi}(\boldsymbol{x},t) := \varphi\left(\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x},t), t\right)$$

$$\boldsymbol{v} := \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \overline{\boldsymbol{u}}^{\mathrm{L}} \longleftarrow \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{\Xi} = \boldsymbol{u}^{\xi}$$

$$(D\varphi/Dt)^{\xi} = \bar{D}^{\mathrm{L}}\varphi^{\xi}$$

$$\bar{D}^{\mathrm{L}} := \frac{\partial}{\partial t} + \bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\nabla$$

$$\longleftarrow \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\varphi^{\xi} = (D\varphi/Dt)^{\xi}$$

Lagrangian mean vs Eulerian mean

Lagrangian mean

$$(D\varphi/Dt)^{\xi} = \bar{D}^{\mathrm{L}}\varphi^{\xi} \quad \Rightarrow \quad \boxed{\begin{aligned}\overline{(D\varphi/Dt)}^{\mathrm{L}} &= \bar{D}^{\mathrm{L}}\overline{\varphi}^{\mathrm{L}} \\ (D\varphi/Dt)^{l} &= \bar{D}^{\mathrm{L}}\varphi^{l}\end{aligned}}$$

exact!

Lagrangian disturbance field $\quad \varphi^{l} := \varphi^{\xi} - \overline{\varphi}^{\mathrm{L}} \quad (\overline{\varphi^{l}} = 0)$

Eulerian mean

Eulerian disturbance field $\quad \varphi' := \varphi - \overline{\varphi} \quad (\overline{\varphi'} = 0)$

$$\overline{(D\varphi/Dt)} = \bar{D}\overline{\varphi} + \overline{\boldsymbol{u}' \cdot \nabla\varphi'},$$
$$(D\varphi/Dt)' = \bar{D}\varphi' + \boldsymbol{u}' \cdot \nabla\overline{\varphi} + \boldsymbol{u}' \cdot \nabla\varphi' - \overline{\boldsymbol{u}' \cdot \nabla\varphi'}$$

Stokes correction $\quad \overline{\varphi}^{\mathrm{S}}(\boldsymbol{x}, t) = \overline{\varphi}^{\mathrm{L}}(\boldsymbol{x}, t) - \overline{\varphi}(\boldsymbol{x}, t)$

$$\overline{\varphi}^{\mathrm{S}} = \overline{\xi_j \varphi'_{,j}} + \frac{1}{2}\overline{\xi_j \xi_k}\,\overline{\varphi}_{,jk} + O(a^3); \quad (\varphi_{,j} := \partial\varphi/\partial x_k$$

Equations of Lagrangian mean field

$$\bar{D}^{\mathrm{L}}\tilde{\rho} + \tilde{\rho}\nabla\cdot\overline{u}^{\mathrm{L}} = 0; \quad \tilde{\rho} := \rho^{\xi}J,\ J := \det\{\Xi_{i,j}\}$$

$$\bar{D}^{\mathrm{L}}\overline{S}^{\mathrm{L}} + \overline{Q}^{\mathrm{L}} = 0,$$

$$\bar{D}^{\mathrm{L}}\left(\overline{u}_i^{\mathrm{L}} - p_i\right) = \overline{(p^l)_{,i}q};$$

$$\rho := F(s,p),\ q := \{F(\overline{S}^{\mathrm{L}}, p^{\xi})\}^{-1} - \{F(\overline{S}^{\xi}, p^{\xi})\}^{-1}$$

$$p_i(\boldsymbol{x},t) := -\overline{\xi_{j,i}u_j^l} \quad \text{pseudomomentum}$$

modelling that respecst topological invariants

use of variational principle:

Euler-Poincare framework

turbulent modelling: LES

Summary

Linear stability of an *circular vortex* subjected to Corilolis force, confined in a cylinder, to three-dimensional disturbances is calculated. This is a parametric resonance instability between two Kelvin waves caused by a perturbation breaking S^1-symmetry of the circular core.

1. Maharov ('93) is simplified; Disturbance field and growth rate are written out in terms of the Bessel and modified Bessel functions.
2. Energetics: Energy of the Kelvin waves is calculated by adapting Cairns' formula (= *black box*) $\longleftrightarrow$ consistent with Krein's theory

3. **Lagrangian approach**: Energy of the Kelvin waves is calculated by restricting disturbance to ***kinematically accessible field***

   linear perturbation is sufficient to calculate energy, quadratic in amplitude!

   $\Rightarrow$ Generation of mean azimutal velocity $\overline{\delta^2 u_\theta}$

4. ***Axial current***: For the *Rankine vortex*, 2 nd-order drift current $\overline{\delta^2 u}$ includes not only azimuthal but also *axial* component $\overline{\delta^2 u_z}$.